Mathematical Programming Computation (2019) 11:703–754
https://doi.org/10.1007/s12532-019-00162-6

FULL LENGTH PAPER

New exact approaches to row layout problems

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Received: 21 February 2018 / Accepted: 10 April 2019 / Published online: 11 May 2019

Abstract

Given a set of departments, a number of rows and pairwise connectivities between these departments, the multi-row facility layout problem (MRFLP) looks for a non-overlapping arrangement of these departments in the rows such that the weighted sum of the center-to-center distances is minimized. As even small instances of the MRFLP are rather challenging, several special cases have been considered in the literature. In this paper we present new mixed-integer linear programming formulations for the (space-free) multi-row facility layout problem with given assignment of the departments to the rows that combine distance and betweenness variables. Using these formulations instances with up to 25 departments can be solved to optimality (within at most 6 h) for the first time. Furthermore, we are able to reduce the running times for instances with up to 23 departments significantly in comparison to the literature. Later on we use these formulations in an enumeration scheme for solving the (space-free) multi-row facility layout problem. In particular, we test all possible row assignments, where some assignments are excluded due to our new combinatorial investigations. For the first time this approach enables us to solve instances with two rows with up to 16 departments, with three rows with up to 15 departments and with four and five rows with up to 13 departments exactly in reasonable time.

A short paper containing some of the results on the parallel row ordering problem and the space-free double-row facility layout problem without proofs appeared in the Proceedings of the OR 2015 [28].

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Keywords Double-row layout problem, Facility layout · Integer programming · Exact solution methods

Mathematics Subject Classification 90C10 · 90C57 · 90C27

1 Introduction

In this paper, we focus on mathematical programming approaches that can certify global optimality of solutions for *Multi-Row Facility Layout Problems* (MRFLP) and variants thereof that are special facility layout problems, see [1,17,39] for recent surveys on MRFLP and classifications of facility layout problems in general. We start with an introduction of the problems considered in this paper and of layout problems related to them.

The multi-row facility layout problem An instance of the MRFLP consists of n departments $\{1,\dots,n\} =: [n]$ with given positive lengths $\ell_i > 0, i \in [n]$, pairwise non-negative connectivities $w_{ij} \geq 0, i, j \in [n], i < j$, and a set $\mathcal{R} := [m]$ of rows available for placing the departments. For sake of simplicity we assume that

- each department can be assigned to any of the given rows,
- inter-row distances between the departments are neglected.

Now the task of the MRFLP is to determine an assignment $r_i \in \mathcal{R}, i \in [n]$, of departments to rows, and feasible horizontal positions for the centers of the departments within the assigned rows, i.e., positions

$$p_i \in \mathbb{R}, i \in [n], \text{ satisf. } \tfrac{1}{2}(\ell_i + \ell_j) \leq |p_i - p_j| \text{ if } r_i = r_j \text{ for a } j \in [n]\backslash\{i\}, \qquad (1)$$

such that the total weighted sum of the center-to-center distances between all pairs of departments is minimized. Hence the MRFLP can be formulated as follows:

$$\begin{aligned} \min \quad & \sum_{\substack{i,j\in[n]\\ i<j}} w_{ij}|p_i - p_j| \\ \text{s. t.} \quad & (r_i = r_j) \Rightarrow \tfrac{1}{2}(\ell_i + \ell_j) \leq |p_i - p_j|, \qquad i, j \in [n], i < j, \\ & r \in \mathcal{R}^n, p \in \mathbb{R}^n. \end{aligned}$$

The MRFLP is of special interest in the design of flexible manufacturing systems because the layout of the machines highly influences the throughput, the material handling time and other performance characteristics of factories [20,36]. Apart from manufacturing planning it has several applications, see e.g., [21,32,55,74] and [43] for more detailed lists. Despite its broad applicability the general MRFLP received only limited attention in the literature. Looking at exact algorithms for the MRFLP we are aware of the following papers. In the 1980s, [37,38] obtained locally optimal solutions for the *Single-Row Facility Layout Problem* (SRFLP) and the *Double-Row Facility Layout Problem* (DRFLP) using a non-linear programming model. The SRFLP, also known as one-dimensional space allocation problem [57], is a special case of the

MRFLP with m equal to 1. The DRFLP is another special MRFLP with $m = 2$, i.e., the departments are arranged above and below a single path. This variant is especially relevant in the planning of factory layouts. Chung and Tanchoco [25] (see also [76]) presented a mixed integer linear programming (MILP) formulation for the DRFLP in 2010. With this approach instances with up to 10 departments could be solved to optimality in 10 min. Later on Amaral [10], again concentrating on the DRFLP, proposed an improved MILP formulation and was able to solve instances with up to 12 departments to optimality. In [3] Amaral solved instances with up to 13 departments using this model and presented an alternative MILP formulation which behaves similarly, but for the larger instances it is usually slower. Based on [10], Secchin and Amaral [68,69] presented an improved MILP allowing them to solve instances with up to 15 departments in less than 5 h (with our computer the solution times are much higher for this model). Recently, Hungerländer and Anjos [43] suggested a semidefinite programming (SDP) approach for the general MRFLP that yields tight global bounds for DRFLP instances with up to 12 departments and MRFLP instances with up to five rows and up to eight departments. Recent works on metaheuristic approaches include, e.g., [27,60,78]. For recent extensions of DRFLP in several directions we refer the reader to [19,71,73,77,78].

Further relevant row layout problems Recent surveys of applications of and global optimization approaches to the SRFLP can, e.g., be found in [15,44,48,50]. Note that in at least one optimal solution of the SRFLP there do not exist spaces between neighboring departments. Hence the SRFLP consists of finding a permutation of the departments that minimizes the total weighted sum of all center-to-center distances. This interpretation is the basis for most heuristic approaches to the SRFLP, see, e.g., [26,35,47,51–54,61,63–65] for some heuristics in the last years. The *Single-Row Equidistant Facility Layout Problem* (SREFLP) [41,62,67] is a special case of the SRFLP with all departments equal in shape. For both SRFLP and SREFLP the largest instances solved to optimality consist of 42 departments [41,44]. Let us also mention that the *Linear Arrangement Problem*, see, e.g., [6,22] for exact solution approaches, where nodes of a graph are assigned to positions on the real number line minimizing the sum of the pairwise distances between adjacent nodes, is a special SREFLP where all connectivities are binary. It is already an NP-hard problem [31], even if the underlying graph is bipartite [30]. For all these single-row variants there exist several types of exact solution approaches. In the earlier papers the models combine binary so called ordering variables that specify the relative position of the departments along the line with big-M-type constraints. These allow to connect the positions of the centers of the departments and the ordering decisions [57]. Most of the best approaches for SRFLP are based on products of ordering variables, which leads to so called betweenness variables specifying whether some department lies between two others. The betweenness models can be divided into the two large groups of (M)ILP and SDP models. Amaral was able to solve instances with up to 35 departments in reasonable time using the ILP model [7], which will be repeated below. The idea of betweenness variables can also be found in [5], but there several constraints restricting the betweenness variables are missing. The so called triplet polytope associated to the model in [7] was further analyzed in [66]. The second large group are SDP models which also handle the product of ordering variables, see [14,16,18]. The best approach in [45] by Hungerländer and Rendl (see also [44]) allows to optimally solve instances with up to 42 departments. Apart from these two groups a lower bounding model based on distance variables is successfully combined with some branch-and-cut scheme in [12].

As the MRFLP (and the DRFLP) is a rather challenging problem and only small instances can be solved to optimality several simplifications have been studied in the literature. In the *Multi-Row Equidistant Facility Layout Problem* (MREFLP) all departments have the same length. Recently, Anjos et al. [13] proposed specially tailored ILP and SDP models for this problem. They could prove global optimality for some instances with up to 25 departments and achieved optimality gaps smaller than 1% for instances with up to 50 departments using a semidefinite programming approach. They also showed how the model specifically tailored to the Double-Row Equidistant Facility Layout Problem, which is related to the quadratic assignment problem (see, e.g., [56]), of Amaral [8] can be improved.

The *Space-Free Multi-Row Facility Layout Problem* (SF-MRFLP) is a restricted version of the MRFLP in which all the rows have a common left origin and empty space between the departments in the same row is not allowed. If we restrict the SF-MRFLP to two rows we obtain the *Space-Free Double-Row Facility Layout Problem* (SF-DRFLP), also denoted as *Corridor Allocation Problem* [9], as a special case. An MILP formulation for the SF-DRFLP is presented in [9]. With this model instances with up to 13 departments can be solved to optimality. We show in the Appendix how this model can be improved using the ideas of [68,69]. Additionally, a semidefinite optimization approach is proposed in [42]. It provides high-quality global bounds for SF-DRFLP instances with up to 15 departments and for SF-MRFLP instances with 3 to 5 rows and 11 departments. Apart from exact approaches, some heuristic approaches have been proposed for SF-DRFLP, see, e.g., [2,49].

Additionally, fixing the row assignment in advance leads to the *k-Parallel Row Ordering Problem* (kPROP) as a simplified SF-MRFLP version (*k* denotes the number of rows in the literature). It asks for an optimal arrangement of the departments along multiple rows, without space between departments in the same row, but where the row of each department is fixed. Hence, the kPROP asks for a permutation of the departments within each row so that the total weighted sum of the center-to-center distances between all pairs of departments (with a common left origin) is minimized. If the kPROP is restricted to two rows we simply call it PROP. Amaral [11] suggested an MILP formulation for the PROP that allowed to solve instances with up to 23 departments to optimality. Furthermore, Hungerländer [40] proposed an SDP approach that yields reasonable global bounds for kPROP instances with up to 100 departments. Some heuristic approaches for PROP were presented in [58].

Illustration The following examples are designed to illustrate the differences between the SRFLP, the kPROP, the SF-MRFLP and the MRFLP. We consider the following instance consisting of 4 departments with lengths $\ell_i = i, i = 1, \ldots, 4$, and pairwise non-zero connectivities $w_{12} = w_{34} = 1, w_{14} = w_{23} = 2$. Figure 1 illustrates optimal layouts for seven different problems in six pictures:

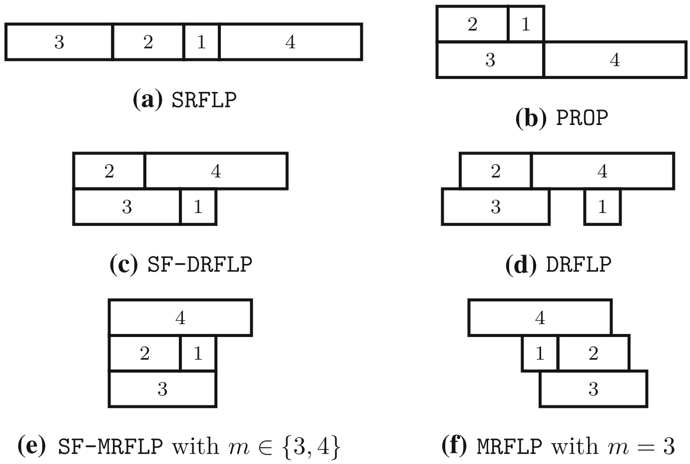

Fig. 1 We consider an instance with $\ell_i = i$, $i = 1, \dots, 4$, $w_{12} = w_{34} = 1$, $w_{14} = w_{23} = 2$, $w_{13} = w_{24} = 0$ and depict optimal layouts for different row layout problems

- Figure 1a shows an optimal layout for the SRFLP with an objective value of $1 \cdot 1.5 + 1 \cdot 6.5 + 2 \cdot 2.5 + 2 \cdot 2.5 = 18$.
- Figure 1b depicts an optimal layout for the PROP when departments 1 and 2 are assigned to row 1 and departments 3 and 4 are assigned to row 2, i.e., $r_1 = r_2 = 1$, $r_3 = r_4 = 2$. The objective value is $1 \cdot 1.5 + 1 \cdot 3.5 + 2 \cdot 2.5 + 2 \cdot 0.5 = 11$.
- Figure 1c shows an optimal layout for the SF-DRFLP with an objective value of $1 \cdot 2.5 + 1 \cdot 2.5 + 2 \cdot 0.5 + 2 \cdot 0.5 = 7$.
- Figure 1d depicts an optimal layout for the DRFLP. The objective value is $1 \cdot 3 + 1 \cdot 3 = 6$. This solution shows that it might be advantageous to have space between neighboring departments in the same row and to allow varying starting points (the leftmost point of the leftmost department in a row) in different rows.
- Figure 1e shows an optimal layout for the SF-MRFLP with $m = 3$ and with $m = 4$ with an objective value of $1 \cdot 1.5 + 1 \cdot 0.5 + 2 \cdot 0.5 + 2 \cdot 0.5 = 4$. This solution shows that enlarging the number of rows does not necessarily lead to a reduction of the weighted sum of distances.
- Figure 1f depicts an optimal layout for the MRFLP with $m = 3$ with an objective value of $1 \cdot 1.5 + 1 \cdot 1.5 = 3$.

Contributions and outline The contributions of this paper are new exact approaches for the MRFLP, the SF-MRFLP, in particular for the DRFLP and the SF-DRFLP, and the kPROP that clearly outperform all exact approaches from the literature for these problems. For the kPROP we transform and extend the semidefinite model from [40] and the ILP models from [7] to an MILP model that yields cheap and strong global bounds compared to other existing models, usually with better root node bounds than the SDP lower bounds in [40]. Especially for PROP instances with rather unbalanced rows, i.e., where one row contains only few departments, we can reduce the running time from days in [11] to less than 1 h. But also for balanced assignments we clearly beat the approach of Amaral in [11]. With our new model we are now able to solve even larger instances than in [11]. For solving the SF-MRFLP and MRFLP exactly we adapt the idea of enumerating over all possible row assignments from the semidefinite approach in [42] to appropriate mixed-integer linear models that are solved to optimality. To further improve the efficiency of the enumeration scheme we prove some combinatorial properties of optimal (space-free) double- and multi-row layouts. So there always exist optimal SF-DRFLP and DRFLP solutions such that both rows are somehow balanced. For SF-MRFLP these results can be used for balancing the two longest rows. Our results can also be used to improve most of the SF-DRFLP and DRFLP models from the literature by a reduction of the associated big-M-values. With our combinatorial results we reduce the number of assignments that have to be considered. Exploiting these improvements our new approach allows us to solve SF-DRFLP and DRFLP instances with up to $n = 16$ for the first time. Additionally we solve SF-MRFLP and MRFLP with three rows for instances with up to $n = 15$ and with four and five rows for instances with up to $n = 13$.

The paper is structured as follows. First, we shortly describe our enumeration scheme. After this we repeat the SRFLP model of Amaral [7], which is later extended to models for kPROP and MRFLP with fixed row assignment. In the main part of Sect. 3 we suggest our new MILP formulation for the kPROP that can be incorporated in an enumerative scheme for solving the SF-MRFLP. Section 4 contains a model for the MRFLP with fixed row assignment that can also be integrated in our enumeration framework. In Sect. 5 we study the structure of optimal solutions of some of the layout problems. These results are later exploited in the computational experiments. Detailed computational results comparing our new solution approaches to others from the literature are reported in Sect. 6 showing the effectiveness of our approaches. Section 7 concludes the paper and gives suggestions for future work.

2 An enumeration scheme for solving row layout problems

In this section we shortly describe our enumeration scheme for solving row layout problems like MRFLP and SF-MRFLP, respectively, exactly. Given an instance of the respective row layout problem, we consider for each possible row assignment the restricted version of the row layout problem, in which the assignment is fixed. Hopefully, these restricted problems are easier to solve because of less degree of freedom. We then enumerate over all possible row assignments in a branch-and-cut approach. Clearly, we can stop a single solution step if the lower bound in branch-and-cut exceeds the objective value of the currently best solution of the general problem. Our enumeration scheme is shown in Algorithm 1. The number of indistinguishable row assignments for the MRFLP was studied in [43]. For the SF-MRFLP the number is usually higher because there might be empty rows. Due to symmetry the number of row assignments that have to be tested is much smaller than m^n. Indeed, we can assume that the sums of the lengths of the departments assigned to each row are non-increasing for increasing row numbers and rows with the same sums of the department lengths can be considered further.

Note that our described approach works well as long as the subproblems with fixed row assignment can be solved quickly for n being not too large. In the following sections we present MILP formulations for the SF-MRFLP and the MRFLP with fixed row assignments to be applied in this enumeration scheme.

Algorithm 1: Enumeration scheme for problem $P \in \{$MRFLP, SF-MRFLP$\}$

Input : instance of P with departments $[n]$, rows $\mathcal{R}$ and connectivities w
Output: optimal value v^* of P

```
1 v* ← ∞
2 for r = (r_1, ..., r_n) ∈ R^n // test all row assignments
  do
     if row assignment r can be neglected then
        continue
     Compute a lower bound v_r (underlined) of P.
     if v_r (underlined) ≥ v* then
        continue
     Determine optimal value v*_r of P with fixed row assignment r.
     if v*_r < v* then
        v* ← v*_r
3 return v*
```

3 An MILP model for the *k*-parallel row ordering problem

In this section we suggest a new model for kPROP. Since this is an extension of the SRFLP model in [7] we start with repeating this.

3.1 A betweenness model for the SRFLP

In the following we shortly repeat the model of Amaral [7] for the SRFLP. Let an SRFLP instance be given with n departments $[n]$ with lengths ℓ_i, $i \in [n]$, and pairwise transports $w_{ij} \geq 0, i, j \in [n], i < j$. By the non-negativity of the pairwise transport weights there always exists an optimal solution such that there do not exist spaces between neighboring departments. So we look for a best permutation of the departments. We introduce binary betweenness variables $\bar{x}_{ijk}, i, j, k \in [n], |\{i, j, k\}| = 3, i < k$, with the interpretation

$$\bar{x}_{ijk} = \begin{cases} 1, & \text{department } j \text{ lies between departments } i \text{ and } k, \\ 0, & \text{otherwise.} \end{cases}$$

Then with constant $C := \frac{1}{2} \sum_{i,j=1,i<j}^{n} w_{ij}(\ell_i + \ell_j)$ Amaral's model reads:

$$\min \sum_{\substack{i,j\in[n]\\ i<j}} w_{ij} \sum_{\substack{k\in[n]\\ k\neq i,j}} \ell_k \bar{x}_{ikj} + C \tag{2}$$

$$\text{s.t. } \bar{x}_{ijk} + \bar{x}_{ikj} + \bar{x}_{jik} = 1, \qquad i,j,k \in [n], i<j<k, \tag{3}$$

$$\bar{x}_{ihj} + \bar{x}_{ihk} + \bar{x}_{jhk} \leq 2, \qquad i,j,k,h \in [n], |\{i,j,k,h\}| = 4, i<j<k, \tag{4}$$

$$\bar{x}_{ihk} - \bar{x}_{ihj} + \bar{x}_{jhk} \geq 0, \qquad i,j,k,h \in [n], |\{i,j,k,h\}| = 4, i<j<k, \tag{5}$$

$$\bar{x}_{ihj} - \bar{x}_{ihk} + \bar{x}_{jhk} \geq 0, \qquad i,j,k,h \in [n], |\{i,j,k,h\}| = 4, i<j<k, \tag{6}$$

$$\bar{x}_{ihj} + \bar{x}_{ihk} - \bar{x}_{jhk} \geq 0, \qquad i,j,k,h \in [n], |\{i,j,k,h\}| = 4, i<j<k, \tag{7}$$

$$\bar{x}_{ijk} \in \{0,1\}, \qquad i,j,k \in [n], |\{i,j,k\}| = 3, i<k. \tag{8}$$

The distance between the centers of two departments $i, j \in [n], i < j$, equals the sum of the lengths of the departments between them plus half the lengths of both departments, see (2). Since all departments are arranged on a line if we consider three department, one of them lies in the middle of these three which is enforced by (3). Furthermore, one ensures via (4)–(7) that the betweenness variables satisfy certain transitivity conditions. After defining the model above, Amaral [7] suggested to further reduce the number of variables and constraints. A polyhedral study of the associated polytope was done in [66].

3.2 A new model for kPROP

In the following we suggest MILP models for the kPROP, where we consider three variants for determining the distances of departments in different rows. We distinguish between the following variants, where d_{ij} denotes the distance between departments $i, j \in [n], i < j$, and p_i denotes the x-coordinate of the center of department $i \in [n]$, its horizontal position:

- Direct variant (this is the standard variant, see, e.g., [11])—type 1: $d_{ij} = |p_i - p_j|$.
- Border variant—type 2: $d_{ij} = \begin{cases} p_i + p_j, & \text{if } r_i \neq r_j, \\ |p_i - p_j|, & \text{otherwise.} \end{cases}$
- Combined variant—type 3: $d_{ij} = \begin{cases} p_i + p_j, & \text{if } |r_i - r_j| \geq 2, \\ |p_i - p_j|, & \text{otherwise.} \end{cases}$

In variant 1 only the distances in x-direction are considered. In variants 2 and 3 this is still true for departments in the same row. Considering type 2 we sum up the distances to the left border for departments in different rows. This can be interpreted in the following way. The means of transport can only move in x-direction and so, for going from one to another row, it first has to go from one of the departments to the left border, then it goes to the other row (for fixed row assignments this length is fixed and so it can be neglected during optimization) and then it moves to the other department. This situation can for instance be found if the transport between the rows is done via some fork lift truck which can only change the rows at the border due to its turning radius. The only difference between type 2 and type 3 is that for type 3 we allow direct connections between neighboring rows. So the fork lift truck might be allowed to

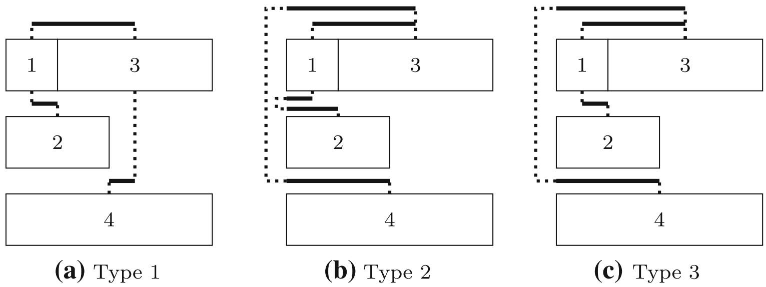

Fig. 2 Visualization of the three distance calculation types. We consider an instance with 4 departments and $\ell_i = i, i = 1, \ldots, 4$, row assignments $r_1 = r_3 = 1, r_2 = 2, r_4 = 3$, and non-zero connectivities $w_{12} = 1, w_{13} = 2$ and $w_{34} = 3$. The objective value for the displayed solution equals $1 \cdot 0.5 + 2 \cdot 2 + 3 \cdot 0.5 = 6$ for distance-calculation type 1, $1 \cdot (0.5+1) + 2 \cdot 2 + 3 \cdot (2.5+2) = 19$ for type 2, and $1 \cdot 0.5 + 2 \cdot 2 + 3 \cdot (2.5+2) = 18$ for type 3

directly move goods between departments in neighboring rows. A second application of distance types 2 and 3 can be found in the design of certain bay storage systems where the transports are carried out by some automated storage system [23,59], e.g. some automated guided vehicle system, which can usually not arbitrarily switch between the rows since it is moving along the aisles. Apart from this, type 2 allows to model the distances for layouts with pier-like structures [24]. Figure 2 shows an illustration of the distance calculation types.

To simplify the presentation of the models we define the sets R_h, $h \in \mathcal{R}$, that contain the indices of the departments assigned to row h, i.e., $j \in R_h \Leftrightarrow r_j = h$. Further we introduce two additional dummy departments $n+1$ and $n+2$, to be placed at the left and the right boundary of each row, respectively, with $\ell_{n+1} = \ell_{n+2} = 0$ as well as $w_{ij} = 0$ if $i, j \in [n+2]$, $\{i, j\} \cap \{n+1, n+2\} \neq \emptyset$. We set $\tilde{R}_h := R_h \cup \{n+1, n+2\}$ for $h \in \mathcal{R}$.

Our idea is now to combine the betweenness model of Amaral [7], repeated in (2)–(8), for each single row (of original departments and the two dummy departments) with distance variables and constraints that couple the decisions on the rows. Instead of adding position variables explicitly, we will measure the distances relative to the dummy departments. Additionally, we will show that certain betweenness variables related to the dummy departments comply with ordering variables. This will allow us to calculate the distances between the departments correctly for all row layout problems considered in the remainder of the paper and to ensure the non-overlapping of departments in the same row.

Our MILP formulations for `kPROP` and `FR-MRFLP` use betweenness variables

$$x_{ikj} = \begin{cases} 1, & \text{if } k \text{ lies between } i \text{ and } j \text{ in the same row,} \\ 0, & \text{otherwise,} \end{cases}$$

for $l \in \mathcal{R}, i, j, k \in \tilde{R}_l, |\{i, j, k\}| = 3, i < j$, and additionally distance variables $d_{ij} \geq 0, i, j \in [n+1], i < j$.

Using these variables we can formulate the kPROP as follows. For all three distance calculation types we use the basic model

$$\min \sum_{\substack{i,j \in [n] \\ i<j}} w_{ij} d_{ij}$$

$$\text{s.t. } x_{ijk} + x_{ikj} + x_{jik} = 1, \qquad l \in \mathcal{R}, i, j, k \in \tilde{R}_l, i < j < k, \tag{9}$$

$$x_{(n+1)i(n+2)} = 1, \qquad i \in [n], \tag{10}$$

$$x_{i(n+1)j} = 0, \qquad l \in \mathcal{R}, i, j \in R_l \cup \{n+2\}, i < j, \tag{11}$$

$$x_{i(n+2)j} = 0, \qquad l \in \mathcal{R}, i, j \in R_l \cup \{n+1\}, i < j, \tag{12}$$

$$x_{ji(n+1)} = x_{ij(n+2)}, \qquad l \in \mathcal{R}, i, j \in R_l, i \neq j, \tag{13}$$

$$+ x_{ihj} + x_{ihk} + x_{jhk} \leq 2, \qquad l \in \mathcal{R}, i, j, k, h \in \tilde{R}_l, |\{i, j, k, h\}| = 4, \; i < j < k, \tag{14}$$

$$- x_{ihj} + x_{ihk} + x_{jhk} \geq 0, \qquad l \in \mathcal{R}, i, j, k, h \in \tilde{R}_l, |\{i, j, k, h\}| = 4, \; i < j < k, \tag{15}$$

$$+ x_{ihj} - x_{ihk} + x_{jhk} \geq 0, \qquad l \in \mathcal{R}, i, j, k, h \in \tilde{R}_l, |\{i, j, k, h\}| = 4, \; i < j < k, \tag{16}$$

$$+ x_{ihj} + x_{ihk} - x_{jhk} \geq 0, \qquad l \in \mathcal{R}, i, j, k, h \in \tilde{R}_l, |\{i, j, k, h\}| = 4, \; i < j < k, \tag{17}$$

$$d_{ij} = \sum_{\substack{k \in R_l \\ k \neq i,j}} \ell_k x_{ikj} + \frac{\ell_i + \ell_j}{2}, \qquad l \in \mathcal{R}, i, j \in R_l \cup \{n+1\}, i < j, \tag{18}$$

$$x_{ijk} \in \{0, 1\}, \qquad l \in \mathcal{R}, i, j, k \in \tilde{R}_l, |\{i, j, k\}| = 3, i < k, \tag{19}$$

$$d_{ij} \geq 0, \qquad i, j \in [n+1], i < j. \tag{20}$$

Equations (9) and inequalities (14)–(17) express that the departments in each row do not overlap and that they correspond to some permutation in each row. Indeed, these constraints are variants of the SRFLP model by Amaral [7] repeated in (2)–(8). So if $i, j, k \in [n]$, $|\{i, j, k\}| = 3$, lie in the same row, exactly one of them lies in the middle [see (9)] and certain transitivity rules have to be satisfied. Furthermore, we ensure by (10)–(13) that in each row all departments lie between the dummy departments $n+1$ (the left border of the layout) and $n+2$ (the right border in each row). The distance of two departments in the same row equals the sum of the lengths of all departments between them plus half the length of both departments, see (18) and compare to the objective (2) of the SRFLP model by Amaral [7]. It remains to ensure that the distances between departments in different rows are calculated correctly. Naturally, these calculations depend on the distance type.

Remark 1 In our model the variables $x_{ji(n+1)}$, $i, j \in [n]$, $i \neq j$, (or $x_{ij(n+2)}$) comply with ordering variables that indicate whether a department lies right or left to a second department, i.e.,

$$x_{ji(n+1)} = x_{ij(n+2)} = \begin{cases} 1, & \text{if } i \text{ lies left to } j \text{ in row } r_i = r_j, \\ 0, & \text{otherwise,} \end{cases}$$

see also models for various row layout problems or the linear ordering problem, e.g., [25,33]. The constraints (9) and (11) imply that $x_{ji(n+1)} = 1 - x_{ij(n+1)}$, which is used in the formulations of classic ordering problems to half the number of variables, see, e.g., [34]. Furthermore note that the 3-cycle inequalities

$$0 \leq x_{ji(n+1)} + x_{kj(n+1)} - x_{ki(n+1)} \leq 1, \;\; l \in \mathcal{R}, i, j, k \in R_l, |\{i, j, k\}| = 3, \quad (21)$$

are implied for three departments i, j, k lying in the same row by (9), (11) and (15)–(17).

For proving this let $l \in \mathcal{R}, i, j, k \in R_l, |\{i, j, k\}| = 3$ and let, w.l.o.g., $i < j < k$. First we show that $0 \leq x_{ji(n+1)} + x_{kj(n+1)} - x_{ki(n+1)}$ is implied. By (15)–(17) we know that

$$\begin{aligned} -x_{ki(n+1)} + x_{ji(n+1)} + x_{jik} &\geq 0, \\ -x_{ij(n+1)} + x_{kj(n+1)} + x_{ijk} &\geq 0, \\ -x_{jk(n+1)} + x_{ik(n+1)} + x_{ikj} &\geq 0. \end{aligned}$$

Summing up and together with $x_{ij(n+1)} + x_{ji(n+1)} = x_{ik(n+1)} + x_{ki(n+1)} = x_{jk(n+1)} + x_{kj(n+1)} = 1$, which is implied by (9) and (11), we get

$$\begin{aligned} -x_{ki(n+1)} + \underbrace{x_{ik(n+1)}}_{1-x_{ki(n+1)}} + x_{ji(n+1)} - \underbrace{x_{ij(n+1)}}_{1-x_{ji(n+1)}} + x_{kj(n+1)} \\ - \underbrace{x_{jk(n+1)}}_{1-x_{kj(n+1)}} + \underbrace{x_{ijk} + x_{jik} + x_{ikj}}_{=1} &\geq 0 \\ \Leftrightarrow \; -2x_{ki(n+1)} + 2x_{ji(n+1)} + 2x_{kj(n+1)} &\geq 0 \\ \Leftrightarrow \; x_{ji(n+1)} + x_{kj(n+1)} - x_{ki(n+1)} &\geq 0. \end{aligned}$$

Second, we prove that $x_{ji(n+1)} + x_{kj(n+1)} - x_{ki(n+1)} \leq 1$ is implied by (9), (11) and (15)–(17). Multiplying each of the following inequalities by -1

$$\begin{aligned} -x_{ji(n+1)} + x_{ki(n+1)} + x_{jik} &\geq 0, \\ -x_{kj(n+1)} + x_{ij(n+1)} + x_{ijk} &\geq 0, \\ -x_{ik(n+1)} + x_{jk(n+1)} + x_{ikj} &\geq 0, \end{aligned}$$

then summing up and using (9) we get

$$\begin{aligned} x_{ji(n+1)} - \underbrace{x_{ij(n+1)}}_{1-x_{ji(n+1)}} - x_{ki(n+1)} + \underbrace{x_{ik(n+1)}}_{1-x_{ki(n+1)}} + x_{kj(n+1)} - \underbrace{x_{jk(n+1)}}_{1-x_{kj(n+1)}} &\leq 1 \\ \Leftrightarrow 2x_{ji(n+1)} + 2x_{kj(n+1)} - 2x_{ki(n+1)} &\leq 2 \end{aligned}$$

$$\Leftrightarrow x_{ji(n+1)} + x_{kj(n+1)} - x_{ki(n+1)} \leq 1.$$

Our model does not contain position variables for each of the departments explicitly as, e.g., in [25], but we can use the following result.

Remark 2 In our model for the kPROP the position (along a horizontal line) of each department $i \in [n]$, i.e., p_i in the problem description (1), is given by $d_{i(n+1)}$. Note that (18) and $\ell_{n+1} = 0$ imply that $d_{i(n+1)}$ equals half the length of department $i \in [n]$ plus the sum of the lengths of all departments between the left border of the layout and i.

We can further strengthen the model by using the clique constraints introduced in [12],

$$\sum_{\substack{i,j \in R_l \\ i<j}} \ell_i \ell_j d_{ij} = \frac{1}{6}\left(\left(\sum_{i\in R_l} \ell_i\right)^3 - \sum_{i \in R_l} \ell_i^3\right), \qquad l \in \mathcal{R}, \tag{22}$$

$$\sum_{\substack{i,j \in S \\ i<j}} \ell_i \ell_j d_{ij} \geq \frac{1}{6}\left(\left(\sum_{i\in S} \ell_i\right)^3 - \sum_{i \in S} \ell_i^3\right), \qquad l \in \mathcal{R}, S \subseteq R_l. \tag{23}$$

Depending on the measurement of the distances, the following inequalities are used for calculating inter-row distances. For distance-calculation type 1, we use

$$d_{ij} \geq d_{i(n+1)} - d_{j(n+1)}, \qquad l, o \in \mathcal{R}, l \neq o, i \in R_l, j \in R_o, i < j, \tag{24}$$

$$d_{ij} \geq d_{j(n+1)} - d_{i(n+1)}, \qquad l, o \in \mathcal{R}, l \neq o, i \in R_l, j \in R_o, i < j, \tag{25}$$

$$\left.\begin{array}{l} d_{ij} + d_{jk} \geq d_{ik}, \\ d_{ik} + d_{jk} \geq d_{ij}, \\ d_{ij} + d_{ik} \geq d_{jk}, \end{array}\right\} \qquad i, j, k \in [n+1], i < j < k, \tag{26}$$

for the distance calculation, where (24) and (25) model the absolute value of the difference of the center positions of $i, j \in [n], i < j$, and (26) are the classic triangle inequalities.

Considering type 2, the distances between departments $i, j \in [n], i \neq j, r_i \neq r_j$, lying in different rows are determined by summing up the distances of each of the two departments to the left border, i.e., to $n + 1$. We model this via

$$d_{ij} = d_{i(n+1)} + d_{j(n+1)}, \qquad l, o \in \mathcal{R}, l \neq o, i \in R_l, j \in R_o, i < j. \tag{27}$$

For distance type 3, we combine the formulas for type 1 and type 2. Indeed, inequalities (24)–(27) are slightly adapted, depending on whether two departments $i, j \in [n], i < j, r_i \neq r_j$, lie in rows that are neighbored or not:

$$d_{ij} \geq d_{i(n+1)} - d_{j(n+1)}, \qquad l, o \in \mathcal{R}, |l - o| = 1, i \in R_l, j \in R_o, i < j, \tag{28}$$

$$d_{ij} \geq d_{j(n+1)} - d_{i(n+1)}, \qquad l, o \in \mathcal{R}, |l - o| = 1, i \in R_l, j \in R_o, i < j, \tag{29}$$

$$d_{ij} = d_{i(n+1)} + d_{j(n+1)}, \qquad l, o \in \mathcal{R}, |l - o| \geq 2, i \in R_l, j \in R_o, i < j, \tag{30}$$

$$\left.\begin{array}{r} d_{ij} + d_{jk} \geq d_{ik}, \\ d_{ik} + d_{jk} \geq d_{ij}, \\ d_{ij} + d_{ik} \geq d_{jk} \end{array}\right\} \qquad \begin{array}{c} i, j, k \in [n], i < j < k, \\ \max\{|r_{l_1} - r_{l_2}| : l_1, l_2 \in \{i, j, k\}\} \leq 1. \end{array} \tag{31}$$

Note that we can only use the triangle inequalities (31) for departments lying in the same row or in two neighboring rows due to the different distance calculations involved.

It is also possible to reduce the number of the distance variables in our models by using Eqs. (18) to replace distance variables for departments assigned to the same row by the sum of betweenness variables. For distance type 2 we do not need the distance variables at all as they can be eliminated with the help of (18) and (27).

Finally, we have a look at all presented models for the different kPROP types.

Remark 3 For all three distance-calculation types the models presented above are indeed formulations of the respective problem. Indeed, using mainly the betweenness model of [7] for each single row [see (2)–(8)] we know that the departments in each row correspond to a feasible ordering. The inner-row distances are calculated by (18) and the absolute value of the difference of the center positions of two departments lying in different rows is modeled via (24) and (25) as well as (28) and (29) (if the connectivities are positive the distance variable $d_{ij}, i, j \in [n], i < j$, equals the distance between departments i and j in all optimal solutions). All other inter-row distances are considered via (27) and (30).

In Sect. 6 the models above are used in the enumerative scheme described in Algorithm 1 to obtain exact approaches for the SF-MRFLP.

4 An MILP model for the MRFLP with fixed row assignment

In this section we propose a model for the MRFLP with fixed row assignment FR-MRFLP. In comparison to the model presented in the last section spaces are allowed between departments lying next to each other in the same row and the rows do not need to have a common left border position. So we now have to ensure that neighboring departments do not overlap and that the distances between the departments are calculated correctly.

Our MILP model for the FR-MRFLP uses the same variables and many constraints of the model for kPROP. In its basic form it is based on distance and betweenness variables, that correspond to ordering variables, see Remark 1. Additionally, we set $M := \sum_{i=1}^{n} \ell_i$. Then our model $\text{IP}_{\text{FR-MRFLP}}$ reads

$$\min \sum_{\substack{i,j \in [n] \\ i<j}} w_{ij} d_{ij}$$

s.t. (9)–(17), (19), (20),

$$d_{j(n+1)} - d_{i(n+1)} \geq M(x_{ji(n+1)} - 1) + \frac{\ell_i + \ell_j}{2}, \qquad l \in \mathcal{R}, i, j \in R_l, i \neq j, \tag{32}$$

$$d_{i(n+1)} \geq \tfrac{\ell_i}{2}, \qquad i \in [n], \tag{33}$$

$$d_{ij} \geq d_{i(n+1)} - d_{j(n+1)}, \qquad i, j \in [n], i < j, \tag{34}$$

$$d_{ij} \geq d_{j(n+1)} - d_{i(n+1)}, \qquad i, j \in [n], i < j. \tag{35}$$

By the considerations from the previous section we know that the betweenness constraints ensure a feasible ordering of the departments (that all lie between departments $n+1$ and $n+2$) in each of the rows. A minimal distance of $\frac{1}{2}(\ell_i + \ell_j)$ between departments $i, j \in R_l \cup \{n+1\}$ assigned to the same row $l \in \mathcal{R}$ is ensured via constraints (32) and (33). So adding the dummy departments and the betweenness variables for them as well allows us the indirect use of ordering variables (special betweenness variables) and position variables (special distance variables), which can be found in the common DRFLP models [10,25]. Inequalities (34) and (35) are special triangle inequalities that bound the distances between the departments. Additionally, we can break some symmetry by setting $x_{ij(n+1)} = 0$ for some fixed pair $|\{i, j\}| = 2$ with $l \in \mathcal{R}, i, j \in R_l$.

Theorem 4 *The model* IP$_{\text{FR-MRFLP}}$ *is a formulation for the* FR-MRFLP*.*

Proof It is well-known that the 3-cycle inequalities on the ordering variables (21), implied by (9), (11) and (15)–(17) according to Remark 1, together with integrality conditions (19) suffice to describe feasible orderings, see, e.g., [33,34,72,75]. The inequalities (32) connect the ordering variables ($x_{ji(n+1)}, i, j \in [n], i \neq j$) with continuous position variables ($d_{i(n+1)}, i \in [n]$) and ensure that all position variables are feasible. Indeed, let a feasible ordering according to the $x_{ji(n+1)}$-variables be given and assume, w. l. o. g., that departments $1, \ldots, h$ lie in row 1 in ascending order. Then the distances $d_{ij}, i, j \in [h], i < j$, satisfy $d_{ij} \geq d_{j(n+1)} - d_{i(n+1)} \geq \frac{\ell_i + \ell_j}{2} + \sum_{k=i+1}^{j-1} \ell_k$ by summing up $d_{j(n+1)} - d_{(j-1)(n+1)} \geq \frac{\ell_j + \ell_{j-1}}{2}, d_{(j-1)(n+1)} - d_{(j-2)(n+1)} \geq \frac{\ell_{j-1} + \ell_{j-2}}{2}, \ldots, d_{(i+1)(n+1)} - d_{i(n+1)} \geq \frac{\ell_{i+1} + \ell_i}{2}$ [see (32)] and (34). (Note, the value of the distance variables d_{ij} might be larger than the actual distance because we modeled the absolute value of the distances between the departments, but assuming positive connectivities the distances have the correct value in all optimal solutions.) □

Replacing the integrality constraints $x_{ijk} \in \{0, 1\}$ by bounds on the betweenness variables $0 \leq x_{ijk} \leq 1$ gives a basic LP relaxation for the FR-MRFLP. To improve the tightness of this relaxation, we can add constraints (26) and (23) introduced above. Equations (18) are not valid anymore, but the position of each department $i \in [n]$, represented by the distance $d_{i(n+1)}$, can be bounded from below by

$$d_{i(n+1)} \geq \sum_{k \in R_l \setminus \{i\}} \ell_k x_{ik(n+1)} + \tfrac{1}{2}\ell_i, \qquad l \in \mathcal{R}, i \in R_l, \tag{36}$$

$$d_{ij} \geq \sum_{k \in R_l \setminus \{i,j\}} \ell_k x_{ikj} + \tfrac{\ell_i + \ell_j}{2}, \qquad l \in \mathcal{R}, i, j \in R_l, i < j. \tag{37}$$

So we can exploit the strength of the betweenness model (see [7]) for departments in the same row and indirectly, by the triangle inequalities, for departments in different rows that are somehow connected via department $n+1$, the left border.

Remark 5 The MRFLP model that we obtain by combining the FR-MRFLP model above and Algorithm 1 can easily be extended to cover further aspects, which might be of practical relevance depending on the application. The main reason for this is that in each step of Algorithm 1 the assignment of the departments to the rows is fixed.

- Smith et al. [70] considered not only the weighted sum of the distances but also the size of a smallest rectangle that contains all departments in the objective function. Given not only the width but also the height of each department this size can easily be calculated or bounded by additional constraints. On the one hand the maximal height in each row is predetermined by the row assignment and on the other hand the length of each row (including the spaces between the departments) equals the maximal distance of the rightmost point of a department in this row to department $n+1$, i.e., to the left border of the layout.
- In our models the inter-row distances are neglected. However, they can easily be handled by our enumeration scheme because they lead to fix costs in the restricted models. In particular, the size of the aisle can be incorporated into the model.
- The model of Chung and Tanchoco [25] (see also [76]) contained minimum clearance conditions, i.e., a minimal distance between two departments if they lie next to each other in the same row. These clearance conditions can easily be included in our model if they satisfy some kind of triangle inequality: the minimal distance between two departments $i, j \in [n], i \neq j$, is not larger than the sum of the minimal distance between i and a third department $k \in [n]\backslash\{i, j\}$, the minimal distance between j and k and the length ℓ_k.

5 Combinatorial properties for speeding up our enumeration

In the following we aim to further improve the computational performance of Algorithm 1. In Sect. 5.1 we do this for the SF-MRFLP by excluding row assignments that are too unbalanced. Later on in Sect. 5.2 we speed up Algorithm 1 for the MRFLP by reducing the big-M value M in (32).

5.1 Excluding unbalanced assignments for the SF-DRFLP and the SF-MRFLP

In the following we show how to speed up Algorithm 1 for the SF-DRFLP and afterwards for the SF-MRFLP by excluding some row assignments. We denote the sum of the lengths of the departments in row $i \in \mathcal{R}$ by $L_i := \sum_{j \in R_i} \ell_j$. Obviously we can restrict attention to all row assignments with $L_i \geq L_j$ $i, j \in \mathcal{R}, i < j$, in Algorithm 1. But further row assignments can be neglected.

We start with the double-row case and distance-calculation type 1. Our aim is to determine the smallest number $g \in \mathbb{R}_+$ such that there always exists an optimal solution of the SF-DRFLP, independent of the objective function, where $g \geq L_1 - L_2$ is satisfied for the corresponding row assignment. Note that in general we have $g \geq \ell_{\max,1} := \max_{i \in R_1} \ell_i$, see Fig. 3 for a small instance where this value for g is attained.

In the following lemma we show that $\ell_{\max,1}$ is an upper bound to g.

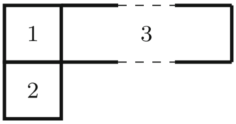

Fig. 3 We consider an instance with $\ell_1 = \ell_2 = 1, \ell_3 = k \geq 2, w_{12} = 4, w_{13} = w_{23} = 1$. In each optimal space-free double-row layout g equals $k = \max_{i \in R_1} \ell_i$ with $R_1 = \{1, 3\}$

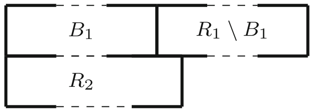

Fig. 4 Illustration of the partition of the departments in row 1

Lemma 6 *For distance-calculation type 1 there always exists an optimal solution (r^*, p^*) of the* SF-DRFLP *with row assignment r^*, where $\ell_{\max,1} \geq L_1 - L_2$ is satisfied for r^*.*

Proof We prove this by contradiction. Let us assume that an optimal space-free double-row layout L^* and a corresponding row assignment r^* are given such that $L_1^* - L_2^* > \ell_{\max,1}$. Furthermore assume that there does not exist an optimal solution L with $L_1 - L_2 \leq \ell_{\max,1}$. Our aim is now to reorder the departments such that afterwards the lengths are somehow balanced and that the objective value is not increased, a contradiction.

We assume, w.l.o.g., that departments $1, \ldots, t$ lie in row 1 and departments $t + 1, \ldots, n$ lie in row 2 in the optimal layout L^* and that these departments are numbered consecutively from left to right in row 1 and from right to left in row 2. Now we define $B_1 := \{i \in R_1 \colon \sum_{j=1}^{i} \ell_j \leq L_2^*\}$ with $|B_1| = s \leq t$. B_1 contains departments $1, \ldots, s$ and the total length of these departments does not exceed L_2^*. We refer to Fig. 4 for an illustration of the departments in B_1.

By our assumption $L_1^* - L_2^* > \ell_{\max,1}$ we know that the set $R_1 \setminus B_1$ contains at least two departments. We suggest a new layout $\hat{L}$ with row assignment $\hat{r}$ that has objective value less than or equal to L^* with the desired property $\hat{L}_1 - \hat{L}_2 \leq \ell_{\max,1}$.

For constructing $\hat{L}$ we first determine $u \in \{s + 1, \ldots, t\}$ minimal such that $\sum_{i=1}^{u-1} \ell_i \geq \sum_{i=u+1}^{n} \ell_i$. This implies $\sum_{i=u}^{n} \ell_i > \sum_{i=1}^{u-2} \ell_i$ and $\sum_{i=1}^{\hat{u}-1} \ell_i \geq \sum_{i=\hat{u}+1}^{n} \ell_i$ for all $t \geq \hat{u} \geq u$. Now in the layout $\hat{L}$ all departments in B_1, R_2 and $B_2 := \{s + 1, \ldots, u - 1\}$ remain at their old position and departments $B_3 := \{u, \ldots, t\}$ are assigned to row 2 in reversed order compared to their order in row 1, i.e., t is assigned right to $t + 1$, $t - 1$ then right to t and so on.

In Fig. 5 we depict L^* and $\hat{L}$. Note, it might happen that now $\hat{L}_2 > \hat{L}_1$, then we have to change the role of row 1 and row 2 later on. But by the choice of u we know $|\hat{L}_1 - \hat{L}_2| \leq \ell_{\max,1}$ by

$$\hat{L}_1 - \hat{L}_2 = \sum_{i=1}^{u-1} \ell_i - \sum_{i=u}^{n} \ell_i = \ell_{u-1} + \underbrace{\sum_{i=1}^{u-2} \ell_i - \sum_{i=u}^{n} \ell_i}_{<0} < \ell_{u-1} \leq \ell_{\max,1},$$

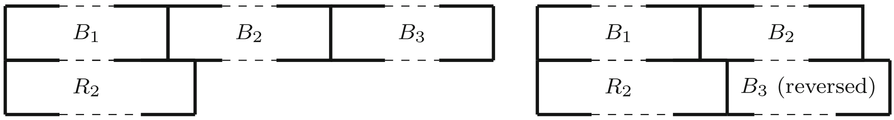

Fig. 5 Comparison of the layouts L^* on the left and $\hat{L}$ on the right

$$\hat{L}_2 - \hat{L}_1 = \sum_{i=u}^{n} \ell_i - \sum_{i=1}^{u-1} \ell_i = \ell_u + \underbrace{\sum_{i=u+1}^{n} \ell_i - \sum_{i=1}^{u-1} \ell_i}_{\leq 0} \leq \ell_u \leq \ell_{\max,1}.$$

Clearly the distances between pairs of departments from $B_1 \cup B_2 \cup R_2$ are the same in L^* and $\hat{L}$. The same is true for pairs of departments from B_3.

- The distances between any department from B_3 and any department from $B_1 \cup R_2$ can only be decreased: On the one hand the centers of the departments from B_3 are still right to the centers of the departments in $B_1 \cup R_2$ due to the definition of B_1. On the other hand the centers of the departments from B_3 are shifted to the left in $\hat{L}$ in comparison to L^*. Indeed, it suffices to consider the position of u. The position of u is $p_u^* = \sum_{i=1}^{u-1} \ell_i + \frac{\ell_u}{2}$ in L^* and $\hat{p}_u = \sum_{i=u+1}^{n} \ell_i + \frac{\ell_u}{2}$ in $\hat{L}$. By the choice of u we have $\hat{p}_u \leq p_u^*$.
- It remains to consider the distances between arbitrary departments $v \in B_2$ and $w \in B_3$ (Fig. 6). The positions are $p_v^* = \sum_{i=1}^{v-1} \ell_i + \frac{\ell_v}{2}$, $p_w^* = \sum_{i=1}^{w-1} \ell_i + \frac{\ell_w}{2}$ in L^* with distance $d_{vw}^* = \sum_{i=v+1}^{w-1} \ell_i + \frac{\ell_w}{2} + \frac{\ell_v}{2}$ and $\hat{p}_v = p_v^*$, $\hat{p}_w^* = \sum_{i=w+1}^{n} \ell_i + \frac{\ell_w}{2}$ in $\hat{L}$. If $\hat{p}_w \geq \hat{p}_v = p_v$, then for the new distance $\hat{d}_{vw}$ between v and w we have $\hat{d}_{vw} \leq d_{vw}$ because w has been moved to the left. If, otherwise, $\hat{p}_w < \hat{p}_v$, then by $v < u \leq w$ and the choice of u we have

$$\begin{aligned}\hat{d}_{vw} - d_{vw}^* &= \sum_{i=w+1}^{n} \ell_i + \tfrac{\ell_w}{2} - \sum_{i=1}^{v-1} \ell_i - \tfrac{\ell_v}{2} - \sum_{i=v+1}^{w-1} \ell_i - \tfrac{\ell_w}{2} - \tfrac{\ell_v}{2} \\ &= \sum_{i=w+1}^{n} \ell_i - \sum_{i=1}^{w-1} \ell_i \leq 0.\end{aligned}$$

In summary, $\hat{L}$ is also an optimal layout, but for the corresponding row assignment $\hat{r}$ the inequality $|\hat{L}_1 - \hat{L}_2| \leq \ell_{\max,1}$ is satisfied. This proves the statement. □

In the equidistant case when all departments have the same length, i.e., for the *Space-Free Double-Row Equidistant Facility Layout Problem* (SF-DREFLP), the result of Lemma 6 further simplifies.

Corollary 7 *For distance-calculation type 1 there always exists an optimal* SF-DREFLP *solution, where*

- *half of the departments are assigned to each of the two rows if n is even and*
- *one row contains $\frac{n+1}{2}$ departments and the other row contains $\frac{n-1}{2}$ departments if n is odd.*

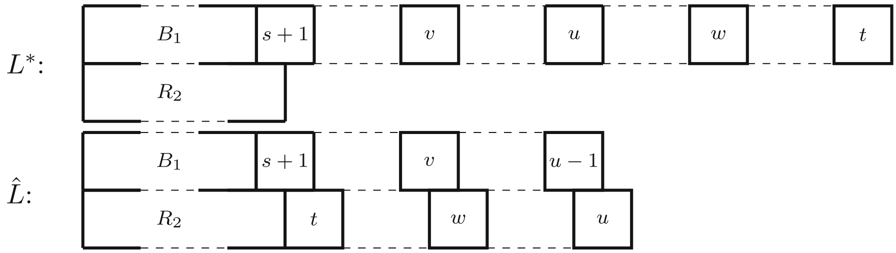

Fig. 6 Illustration of the reduction of the distances between department $v \in B_2$ and department $w \in B_3$. Note that the argument also works for the case $v = u - 1$ and/or $w = u$

For solving the SF-DREFLP we can also use an approach presented in [13]. In order to handle the spaces between departments the authors introduced dummy departments that were added to a DREFLP instance. With the help of these dummy departments the DREFLP reduced to a SF-DRFLP. Hence for the SF-DREFLP with even n we can directly apply the approach from [13] because then both rows have the same length and there do not occur spaces. For the SF-DREFLP with odd n, there is a space of length 1 at the end of row 2 and thus exactly one dummy department is needed. The condition that the dummy department has to be the last department on row 2 can easily be included in the different models from [13]. For instance one can add a constraint guaranteeing that the dummy department does not lie between other departments.

We can also extend the results above to the multi-row case. For this, note that the proof above does not depend on the number of rows m. Hence we can formulate the following corollary.

Corollary 8 *For distance-calculation type 1 the value $\ell_{\max,1}$ is the smallest g such that there always exists an optimal solution to the SF-MRFLP with $m \geq 2$, where $L_i \geq L_j, i, j \in \mathcal{R}, i < j$, and $g \geq L_1 - L_2$ for the corresponding row assignment r^*.*

Remark 9 Note that Corollary 8 is not true for distance-calculation type 2 with $m \geq 2$ and distance-calculation type 3 with $m \geq 3$: For these cases it might be preferable to arrange two departments next to each other in the same row instead of putting them below each other in two different (for type 3 non-neighboring) rows because then the distance of the departments equals the sum of the distances of both departments to the left border of the layout.

In the following two examples we want to show that in general the difference between the row lengths of arbitrary rows cannot be bounded.

Let us first consider an instance with $n \geq 2$ departments such that $\ell_1 = 1$ and $\ell_i = \frac{1}{2n}$ for all $i \in [n] \backslash \{1\}$ and non-zero weight $w_{12} = 1$. Then all optimal kPROP solutions with $m \geq 2$ have the following structure. Department 1 is assigned to one row and all other departments are assigned to a second row where department 2 is the rightmost one in that row. So $m - 2$ rows are empty. For an illustration of an optimal layout we refer to Fig. 7.

Secondly, we demonstrate that the differences of the row lengths $L_i - L_{i+1}$, $i \in \{2, \dots, m-1\}$, can be arbitrarily large in optimal multi-row layouts. To do so we

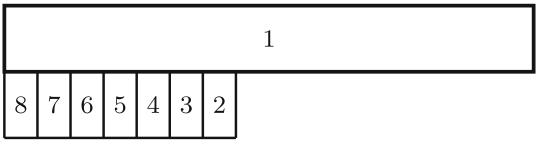

Fig. 7 Illustration of an optimal space-free multi-row layout for an instance with $n = 8, \ell_1 = 1, \ell_2 = \cdots = \ell_8 = \frac{1}{16}$, non-zero weight $w_{12} = 0$ and number of rows $m \geq 2$. Allowing to use more than two rows does not change the structure of optimal solutions as well as the optimal objective value

consider the following instance for fixed $k \in \mathbb{N}$, where we assume, w.l.o.g., that k is even. Here each department is denoted by a tuple (i, j), with $i \in [m-1]$, and $j \in [k(m-2)]$ if $i = 1$ and $j \in [k(m-i)]$ otherwise. We collect all tuples of the departments in the set D. The department lengths and connectivities are chosen as follows:

$$\ell_{i,j} = 1 + 2^{j-1}\varepsilon, \qquad (i, j) \in D, \varepsilon \ll 1, \tag{38a}$$

$$w_{(i,j),(i+1,j)} = 1, \qquad (i, j), (i+1, j) \in D. \tag{38b}$$

One can easily check that in all optimal space-free layouts we have $L_i - L_{i+1} > k$, $i \in \{2, \ldots, m-2\}$, for arbitrary k. For an illustration of the optimal layout structure attaining objective value 0 we refer to Fig. 8.

Note that this example also shows that Corollary 8 is tight in the following sense: We cannot bound the difference of row lengths in space-free layouts in any reasonable way except for the two longest rows.

5.2 Reducing the big-M value *M* in the formulation of the FR-DRFLP

In order to improve our enumeration scheme for the DRFLP we will show that the big-M-value M in $\text{IP}_{\text{FR-DRFLP}}$ and more generally in any DRFLP model can be chosen smaller than $\sum_{i=1}^{n} \ell_i$, which is usually used in the literature, see, e.g., [10,25].

We again start with the double-row case and distance-calculation type 1. Let a feasible DRFLP layout be given and let us assume, w.l.o.g., that the leftmost department i of the considered layout starts at position 0 with its center at $p_i = \frac{\ell_i}{2}$ and that the rightmost department j of this layout finishes at t with its center at $p_j = t - \frac{\ell_j}{2}$. Clearly there exists an optimal layout that does not have space on both rows at any position p with $0 \leq p \leq t$. Now we want to give a bound on t such that there always exists an optimal double-row layout just using the interval $[0, t]$. A straightforward bound for t is $\sum_{i=1}^{n} \ell_i$. In the following lemma we deduce a tighter bound for t.

Lemma 10 *Given a* DRFLP *instance that satisfies, w.l.o.g.,* $\ell_i \leq \ell_{i+1}, i \in [n-1]$, *there always exists an optimal double-row layout on the interval* $[0, t]$ *with*

$$t \leq \sum_{i=\left\lfloor \frac{n+1}{3} \right\rfloor + 1}^{n} \ell_i \tag{39}$$

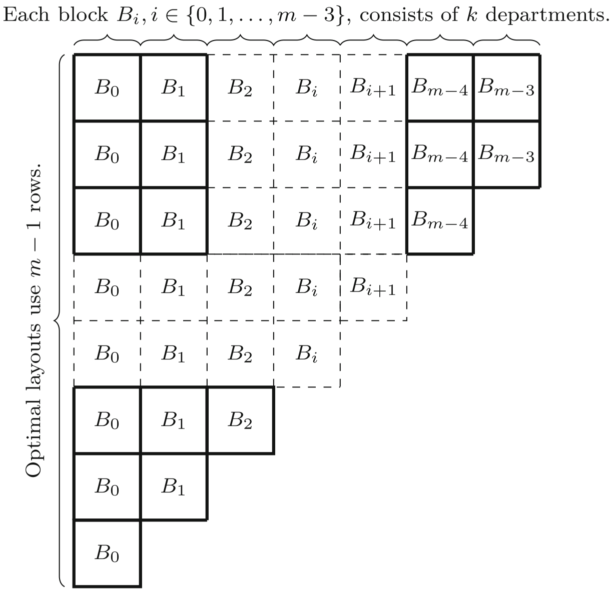

Fig. 8 Illustration of the structure of an optimal space-free multi-row layout for the instance given in (38). The blocks $B_i, i \in \{0, 1, \ldots, m-3\}$, consist of k departments with lengths $1+2^{ik}\varepsilon, \ldots, 1+2^{((i+1)k-1)}\varepsilon$. Row m is empty

for distance-calculation type 1.

Of course this result can be used as an upper bound for the length of rows for MRFLP as well.

Proof For a given solution we denote by $l_i, r_i \in \mathbb{R}$ the left and right end points of department $i \in [n]$, respectively. Consider a tightest optimal solution, i. e., t is as small as possible. This layout does not contain a simultaneous gap on both rows at some position (between departments). Let $D \subseteq [n]$ be a smallest set of departments such that

$$\bigcup_{i \in D} [l_i, r_i] = \bigcup_{i \in [n]} [l_i, r_i] = [0, t].$$

Suppose, for a contradiction, that the tightest optimal solution does not satisfy (39), then $|D| > n - \lfloor \frac{n+1}{3} \rfloor$. We denote by $\mathcal{I}$ the set of intersection intervals of two departments of D, i. e.

$$\mathcal{I} := \{[l_j, r_i] \colon l_i < l_j \leq r_i < r_j, i, j \in D\}.$$

Note that by definition $|\mathcal{I}| = |D| - 1$.

Let $D' := [n] \backslash D$, then $|D'| \leq \lfloor \frac{n+1}{3} \rfloor - 1$. Because D is minimal each department $i \in D'$ intersects with at most two intervals $I_1, I_2 \in \mathcal{I}$ with $[l_i, r_i] \cap I_1 \neq \emptyset$ and

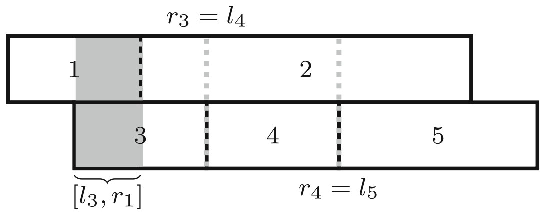

Fig. 9 Visualization of the proof of Lemma 10. Choosing $D = \{1, 3, 4, 5\}$ we can reduce the cardinality of D because department 2 has a non-empty intersection with the 3 intersection intervals $[l_3, r_1]$, $[l_4, r_3]$, $[l_5, r_4]$ (gray part) of the elements in D. The departments $\overline{D} = \{1, 2, 5\}$ with $|\overline{D}| < |D|$ cover the whole interval $[l_1, r_5]$, so D is not a smallest set with this property

$[l_i, r_i] \cap I_2 \neq \emptyset$ (otherwise at least two departments in D could be replaced by i), see Fig. 9. Therefore, because

$$2|D'| \leq 2\left\lfloor \frac{n+1}{3} \right\rfloor - 2 \leq \left\lceil \frac{2n-4}{3} \right\rceil < \left\lceil \frac{2n-1}{3} \right\rceil$$
$$= n - \left\lfloor \frac{n+1}{3} \right\rfloor \leq |D| - 1 = |\mathcal{I}|,$$

there is at least one intersection interval $[l_{\hat{j}}, r_{\hat{\imath}}] \in \mathcal{I}, \hat{\imath}, \hat{j} \in D$, with $[l_{\hat{j}}, r_{\hat{\imath}}] \cap [l_k, r_k] = \emptyset$ for all $k \in D'$. But this implies that there is space left of $\hat{j}$ and right of $\hat{\imath}$ (possibly on the other row), so the layout could be condensed by shifting $\hat{j}$ and all departments to the right of $\hat{j}$ slightly to the left (possibly switching rows first), see Fig. 10. □

The following example shows that the bound in Lemma 10 is tight. We are given an instance with n such that $\mathrm{mod}(n, 3) = 1$ and

$$\ell_i = \begin{cases} 1 + \varepsilon, & i \in [n], \mathrm{mod}(i, 3) = 2, \\ 1, & i \in [n], \mathrm{mod}(i, 3) \neq 2, \end{cases}$$
$$w_{ij} = 1, \quad i \in [n], \mathrm{mod}(i, 3) = 2, \ j \in \{i-1, i+1, i+2\}.$$

In the optimal double-row layout, depicted in Fig. 11, the longer row of the optimal layout has length $n - \lfloor \frac{n+1}{3} \rfloor = n - \frac{n-1}{3}$. The bound on the maximal length t according to (39) is $n - \frac{n-1}{3} + \frac{n-1}{3} \cdot \varepsilon$, so the bound is tight.

Remark 11 The upper bound t on the maximal horizontal length of the double-row layout cannot only be used for improving our MILP model by setting M to a smaller value, but it can also be applied to reduce the number of row assignments that have to be considered in our enumeration scheme as we can neglect all assignments that are unbalanced, i.e., where the sum of the lengths of the departments in one of the rows exceeds t.

Finally, we want to note that Lemma 10 cannot be straightforwardly generalized to the multi-row case using $m \geq 3$ rows and hence we leave the deduction of possible even smaller values of t for larger m for future research. Furthermore Lemma 10 is in

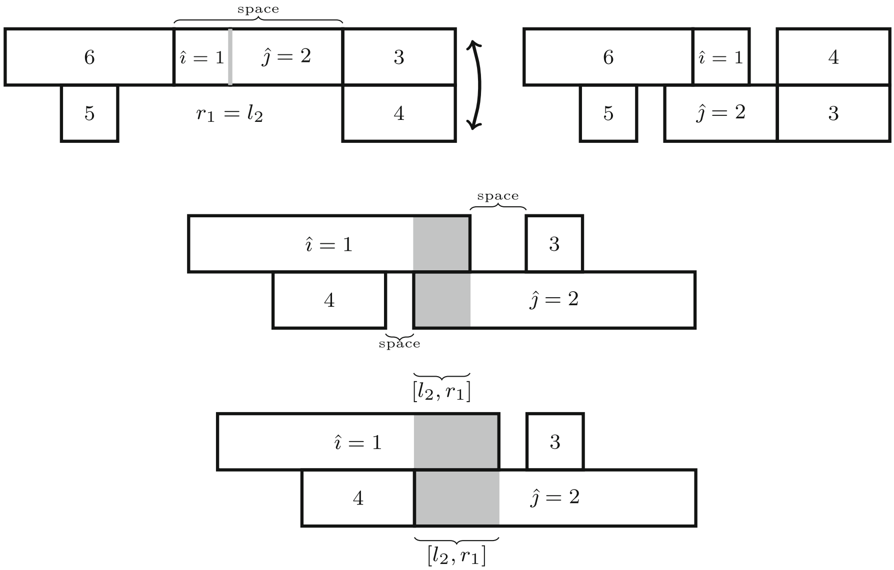

Fig. 10 Visualization of the shifting of departments in the proof of Lemma 10. In the first example we use $D = \{6, \hat{\imath} = 1, \hat{\jmath} = 2, 3\}$ and shift departments 2, 3, 4 after switching the rows. The two departments $\hat{\imath} = 1$, $\hat{\jmath} = 2$ only overlap on the boundary (gray line). In the second example below we use $D = \{\hat{\imath} = 1, \hat{\jmath} = 2\}$. Departments $\hat{\imath} = 1$, $\hat{\jmath} = 2$ overlap (gray part), but right and left there is some space. So we shift departments 2, 3 to the left without enlarging the distance between two arbitrary departments and so that afterwards the departments in each row do not overlap

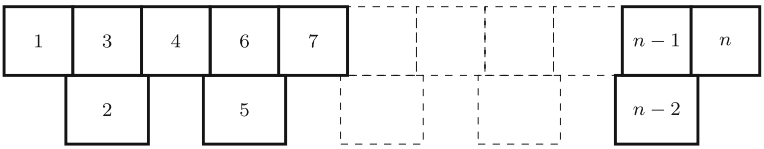

Fig. 11 We are given a DRFLP instance with lengths $\ell_i = 1 + \varepsilon$ for $i \in [n]$, $\mathrm{mod}(i, 3) = 2$, and $\ell_i = 1$ for $i \in [n]$, $\mathrm{mod}(i, 3) \neq 2$, and non-zero connectivities $w_{ij} = 1, i \in [n], \mathrm{mod}(i, 3) = 2, j \in \{i-1, i+1, i+2\}$. For all optimal double-row layouts the bound on t in Lemma 10 is tight

general not true for distance-calculation type 2 because it might again be preferable to arrange two departments next to each other in the same row instead of putting them below of each other in two different rows. If we considered inter-row distances (e.g. modeling the size of the aisle) in our enumeration scheme then none of the combinatorial results determined in this section would apply anymore. The same is true in the multi-row case.

6 Computational experiments

In this section we present the computational results for several PROP, kPROP, SF-MRFLP and MRFLP instances. We used the larger instances from [11] for the PROP and the instances from [40] for the kPROP. Furthermore, we tested all instances used by Amaral in [9] as well as the larger instances from [3] for the SF-DRFLP and [10] for the DRFLP, respectively. Apart from these instances, we created new ones according to the construction schemes in [10,11,40]. For the SF-MRFLP and the MRFLP we only tested these instance classes for n up to 15 for $m = 3$ and for n up to 13 for $m \in \{4, 5\}$. All instances and our source code can be downloaded from [29]. All experiments were performed on a QUAD-Core INTEL-Core-I7-4770 (4×3400 MHz) with 32 GB RAM in single processor mode. We used CPLEX 12.8 [46] as an IP solver. For all different problems except for the multi-row case with $m \geq 3$ we tested two versions. In Full we included all inequalities at once, and in Cuts we only used some of the constraints at once and separated the respective variant of the triangle inequalities (see (26)) and inequalities (14)–(17). For the PROP we also considered the effect of adding the clique constraints (22) and (23), leading to Full-C and Cuts-C. Some preliminary tests, not contained in this paper, indicated that CPLEX should be forced to use all detected violated inequalities until the end. All running times are given in seconds and we used a time limit of 6 h for the PROP and the kPROP instances as well as of 12 h for the SF-DRFLP and DRFLP instances. For the SF-MRFLP and the MRFLP with $m \geq 3$ we only tested Cuts and all additional row assignment reductions and the best big-M value are used.

6.1 Results for the PROP and the kPROP

Considering the PROP instances in Table 1, we see that in comparison to [11] the running times could be reduced significantly. This can be seen especially for instances where only few departments are contained in the first row. These instances were extremely hard when treated with the formulation of Amaral [11]. The average running time for instances with $n = 23$ and only 4 departments in row 1 was 344,869 s in [11] on an Intel Core i3-M330 (2.13 GHz) with 4 GB RAM using CPLEX 12.4. With the fastest variant we solved all these instances in less than 10 min. But also for more balanced row assignments our approach performed much better, allowing us to solve all instances with $n = 25$ in at most 4000 s with the best variant.

Comparing the results of Full and Cuts one can see that using separation often paid off if n was rather large. But there are instances with smaller n and balanced row assignments where Full was better than Cuts. One reason for this effect is that the number of the constraints highly depends on the assignment. For each row $i \in \mathcal{R}$ the number of constraints (9) grows cubic in the size $|R_i|$ and with a power of 4 in $|R_i|$ for constraints (14)–(17). So the number of constraints is much smaller for balanced row assignments than for unbalanced ones. Using additionally the clique constraints was advantageous for some instances (see especially AV25_$*$), but in general there was not a clear winner comparing the four different solution variants. Here we also want to note that in our tests the solution time was partially highly influenced by the time and the quality of the (upper bound) solutions found by CPLEX during the solution process. The optimal values for all instances can be found in Table 2.

The results for the kPROP are presented in Tables 3, 4 and 5. They show that the type of the distance calculation has a large effect on the running time. It seems that instances with distance-calculation type 2 are rather easy. Using Cuts all instances

Table 1 Running times in seconds or gaps (after a time limit TL of 6 h) for PROP instances with $\lfloor \frac{n}{i} \rfloor$ departments in row 1

Name	Source	Full				Cuts				Full-C				Cuts-C			
		$i=2$	$i=3$	$i=4$	$i=5$	$i=2$	$i=3$	$i=4$	$i=5$	$i=2$	$i=3$	$i=4$	$i=5$	$i=2$	$i=3$	$i=4$	$i=5$
P16a	[11]	16.37	28.33	11.41	21.55	13.09	17.78	14.16	14.32	19.36	31.30	12.42	21.74	10.50	17.34	18.12	12.90
P16b	[11]	10.30	16.26	16.22	9.90	12.39	14.28	12.35	13.35	8.90	17.22	14.85	10.50	11.86	14.34	13.19	6.91
P20a	[11]	635.72	1001.57	313.89	157.98	522.52	299.23	282.64	94.26	415.59	358.46	678.18	157.78	528.99	229.20	170.38	126.24
P20b	[11]	243.08	151.77	209.13	200.17	189.63	338.39	166.39	199.24	265.59	214.52	145.75	184.17	161.35	346.01	246.95	141.18
P21a	[11]	411.77	460.00	192.36	360.59	503.34	343.40	257.84	389.65	522.68	716.92	123.21	455.77	461.02	365.52	247.74	485.91
P21b	[11]	392.06	580.29	304.76	168.58	169.22	279.86	157.64	64.88	295.28	514.52	238.22	164.67	117.75	378.54	184.10	89.48
P21c	[11]	269.30	1102.78	316.29	505.95	239.00	1051.62	199.41	246.99	243.08	1085.53	204.87	470.63	199.58	1030.12	164.70	266.55
P21d	[11]	546.31	287.16	320.52	302.74	499.04	185.60	237.81	101.57	236.85	297.98	494.04	286.52	250.78	242.42	197.26	139.31
P21e	[11]	910.22	325.57	263.41	111.10	1070.79	281.00	216.42	76.14	971.17	504.64	198.90	117.19	1728.74	345.49	90.22	81.20
P22a	[11]	501.24	630.20	585.11	793.60	960.69	811.81	467.57	335.98	1332.25	716.52	411.93	879.32	791.45	927.92	300.56	393.02
P22b	[11]	690.63	1303.05	401.12	270.27	1075.30	903.93	350.39	121.22	443.75	1096.48	499.09	328.16	855.54	808.98	212.29	114.71
P22c	[11]	4112.46	1087.20	359.87	806.99	1564.63	763.76	373.28	188.32	1205.57	2042.97	516.86	856.26	1898.95	1277.45	370.90	191.61
P22d	[11]	749.05	1352.79	301.22	181.36	561.56	716.12	84.94	98.49	1051.58	1820.14	345.58	209.70	1019.54	883.71	121.12	81.06
P22e	[11]	738.62	1023.82	1113.51	631.99	1609.46	894.90	537.90	142.22	1192.32	1939.60	915.12	485.98	1039.45	2319.08	477.24	194.79
P23a	[11]	826.50	1745.50	969.55	1152.41	2762.54	1459.40	498.59	1544.12	929.06	1556.44	1100.04	2147.22	1651.61	1145.68	649.48	1092.81
P23b	[11]	544.95	2187.83	579.32	655.61	450.29	862.21	404.35	135.90	525.90	1666.88	355.44	641.73	536.36	1357.25	361.86	144.70
P23c	[11]	1053.54	839.20	623.74	884.28	389.80	574.82	174.25	305.04	563.44	656.67	694.72	937.10	266.38	435.83	188.93	291.28
P23d	[11]	1701.96	1602.51	2963.30	1660.84	957.69	1047.44	359.92	305.84	1095.56	2692.70	1655.52	1542.66	1089.92	1865.10	583.20	698.48
P23e	[11]	1179.25	862.23	361.09	193.81	1003.21	333.69	153.01	74.72	728.98	1069.17	415.96	240.69	954.00	311.19	158.96	79.42

Table 1 continued

Name	Source	Full				Cuts				Full-C				Cuts-C			
		$i=2$	$i=3$	$i=4$	$i=5$	$i=2$	$i=3$	$i=4$	$i=5$	$i=2$	$i=3$	$i=4$	$i=5$	$i=2$	$i=3$	$i=4$	$i=5$
P24a	New	3439.99	2048.33	2165.06	4309.91	12,310.49	1706.36	1075.20	741.26	3350.22	4362.64	1609.09	3146.40	14,992.32	3326.82	1380.72	2338.36
P24b	New	1327.24	3354.56	3113.97	2002.51	1666.85	3387.67	1844.92	590.11	1249.58	3557.51	5331.40	2208.08	1527.07	4216.90	1906.86	589.59
P24c	New	1643.54	3833.98	4537.58	6099.41	3948.58	4614.28	1102.66	1020.50	1124.43	4055.34	3701.35	4247.61	18,277.35	2039.24	1484.40	620.78
P24d	New	1269.41	2872.72	2303.19	2070.42	639.78	1205.49	840.79	733.28	1504.51	1828.90	3496.22	1599.61	840.30	1327.16	1086.10	907.54
P24e	New	869.59	6430.22	2652.25	502.50	991.39	2863.05	1246.54	85.32	867.98	6311.52	2032.60	447.85	1150.31	3541.86	844.56	99.85
P25a	New	5530.93	4503.48	5587.04	5730.18	11,161.20	4456.01	1952.33	1295.45	2991.47	4939.95	4911.80	4984.76	9794.42	3741.52	2684.88	1127.02
P25b	New	4857.82	8675.31	7368.07	7280.56	5242.57	13,769.24	1867.84	2144.20	2804.80	15,146.90	15,440.05	5780.77	5994.68	7859.48	1991.55	2545.58
P25c	New	3381.79	6917.26	6566.56	5261.82	3681.50	9468.46	1630.05	870.37	3274.73	8473.39	8328.47	4906.89	5863.98	6787.59	2629.75	784.57
P25d	New	2703.64	6235.17	1826.44	1812.07	1084.81	3341.93	911.73	477.26	2349.31	3061.23	2364.68	2877.70	2647.95	3261.04	873.92	635.06
P25e	New	1940.50	3349.69	8537.12	9951.68	2770.02	2199.35	3425.80	2173.24	4082.60	3610.54	6294.98	3312.69	5320.82	2356.02	2374.26	2027.53
AV25_1	[16]	TL (0.02)	TL (0.03)	7141.59	3204.66	TL (0.01)	8226.46	1111.77	417.32	TL (0.00)	TL (0.02)	8410.00	2387.22	18,046.46	5980.89	1183.30	492.69
AV25_2	[16]	19,020.00	3857.62	4004.21	2222.80	7323.16	2011.48	1372.32	962.42	7009.52	5060.30	5624.66	1692.26	7039.98	1240.18	1413.09	693.99
AV25_3	[16]	TL (0.01)	2701.16	2031.98	2365.87	19,245.30	923.03	975.73	520.44	3158.89	3043.12	3100.42	2874.21	18,714.69	1152.37	864.67	601.78
AV25_4	[16]	5230.91	2572.96	6634.65	1625.38	7383.12	1426.93	1705.29	971.61	4437.43	2234.07	5219.47	2880.89	6972.88	1673.60	1851.56	1009.63
AV25_5	[16]	3763.89	5019.78	2717.70	1588.78	14,422.67	1284.40	1487.65	1397.27	3874.38	4537.13	2357.62	1285.87	6683.53	1616.57	2567.98	843.65

In `Full-C` and `Cuts-C` we also add the clique constraints (22) and (23) to our basic model (9)–(20)

Table 2 Optimal values of PROP instances with $\lfloor \frac{n}{i} \rfloor$ departments in row 1

Name	Source	$i = 2$	$i = 3$	$i = 4$	$i = 5$
P16a	[11]	7630.0	9813.0	11,409.0	12,279.0
P16b	[11]	6239.5	9091.5	9636.5	11,256.5
P20a	[11]	12,609.5	15,874.5	18,185.5	21,215.5
P20b	[11]	12,936.0	19,167.0	22,801.0	23,902.0
P21a	[11]	7006.5	9141.5	11,765.5	12,382.5
P21b	[11]	11,705.0	13,887.0	18,564.0	20,825.0
P21c	[11]	11,434.0	12,758.0	16,888.0	19,481.0
P21d	[11]	12,289.0	14,988.0	19,471.0	20,685.0
P21e	[11]	13,112.5	15,711.5	19,865.5	22,423.5
P22a	[11]	8874.0	12,238.0	15,385.0	16,114.0
P22b	[11]	15,714.0	19,183.0	23,534.0	25,044.0
P22c	[11]	14,693.0	19,963.0	24,221.0	25,545.0
P22d	[11]	16,355.0	19,981.0	25,180.0	26,796.0
P22e	[11]	14,815.5	20,112.5	24,515.5	27,161.5
P23a	[11]	10,242.0	14,294.0	17,812.0	18,619.0
P23b	[11]	15,802.5	21,116.5	26,004.5	29,892.5
P23c	[11]	15,542.0	21,511.0	26,040.0	27,553.0
P23d	[11]	17,174.0	23,522.0	27,922.0	30,694.0
P23e	[11]	16,481.5	20,798.5	27,574.5	29,810.5
P24a	New	11,778.0	14,730.0	18,757.0	21,729.0
P24b	New	17,629.5	21,015.5	25,311.5	29,881.5
P24c	New	17,378.5	18,630.5	23,909.5	29,041.5
P24d	New	19,630.0	21,786.0	26,949.0	32,132.0
P24e	New	21,400.0	26,998.0	33,235.0	38,800.0
P25a	New	12,889.5	16,411.5	20,865.5	22,990.5
P25b	New	17,459.0	23,635.0	27,162.0	29,496.0
P25c	New	23,148.5	33,228.5	40,086.5	41,264.5
P25d	New	22,421.0	29,450.0	35,283.0	38,373.0
P25e	New	21,048.5	24,664.5	32,078.5	34,018.5
AV25_1	[16]	2349.0	3077.0	3705.0	4039.0
AV25_2	[16]	19,138.5	23,826.5	26,229.5	30,193.5
AV25_3	[16]	12,549.0	18,714.0	23,081.0	23,167.0
AV25_4	[16]	24,922.5	30,647.5	33,584.5	38,689.5
AV25_5	[16]	8011.0	10,126.0	11,289.0	12,951.0

could be solved to optimality in less than one second. This was even much faster than the running times in [40] where one only gets lower bounds. Using type 3, the running times were a bit higher, but we also only needed at most 4 min. Similarly as for PROP, there was not a clear winner between Full and Cuts for distance-calculation type 1. For the smaller instances Cuts is often better, but for $n \geq 22$ the running times of both methods highly depend on the instance and the chosen number of rows.

Table 3 Running times in seconds and optimal values "opt" for `kPROP` instances from [40] with $n \in \{16, 20, 21\}$ with $\lfloor \frac{n}{i} \rfloor$ departments in each of the first $m-1$ rows

Name	m	i	`Full1`	`Full2`	`Full3`	`Cuts1`	`Cuts2`	`Cuts3`	opt 1	opt 2	opt 3
P16a	3	3	4.63	0.01	1.10	5.02	0.00	1.30	5234.0	13,936.0	7910.0
P16a	3	4	3.76	0.02	1.10	2.87	0.00	1.22	6639.0	13,541.0	9615.0
P16a	3	5	2.75	0.08	1.01	2.71	0.01	1.05	9746.0	14,843.0	12,845.0
P16a	4	4	1.67	0.00	0.14	1.12	0.00	0.18	3896.0	10,883.0	7191.0
P16a	4	5	2.96	0.01	0.30	1.58	0.00	0.39	6021.0	12,370.0	9630.0
P16a	5	5	1.30	0.00	0.02	0.59	0.00	0.02	3181.0	9412.0	7073.0
P16b	3	3	2.34	0.01	0.60	1.71	0.00	0.83	4289.5	11,375.5	6330.5
P16b	3	4	2.10	0.02	0.92	2.06	0.00	1.03	5228.5	11,123.5	7583.5
P16b	3	5	4.06	0.08	1.09	4.53	0.01	0.88	9123.5	12,644.5	10,981.5
P16b	4	4	1.26	0.00	0.16	0.71	0.00	0.26	3401.5	9028.5	5993.5
P16b	4	5	1.87	0.01	0.28	1.18	0.00	0.30	4717.5	9834.5	7299.5
P16b	5	5	0.83	0.00	0.07	0.50	0.00	0.07	2786.5	7688.5	5934.5
P20a	3	3	125.86	0.03	3.45	127.69	0.00	4.46	8587.5	21,440.5	13,317.5
P20a	3	4	63.94	0.07	10.51	94.06	0.01	10.17	11,223.5	22,952.5	16,161.5
P20a	3	5	70.59	0.16	15.80	136.80	0.02	16.95	14,171.5	23,096.5	18,129.5
P20a	4	4	27.83	0.00	1.26	35.09	0.00	1.27	6697.5	17,663.5	12,064.5
P20a	4	5	27.10	0.02	0.69	13.64	0.00	0.92	7653.5	17,694.5	13,524.5
P20a	5	5	7.74	0.00	0.12	3.67	0.00	0.11	5210.5	14,317.5	10,532.5

Table 3 continued

Name	m	i	`Full1`	`Full2`	`Full3`	`Cuts1`	`Cuts2`	`Cuts3`	opt 1	opt 2	opt 3
P20b	3	3	38.38	0.03	3.51	43.17	0.00	4.28	9396.0	22,750.0	14,073.0
P20b	3	4	25.35	0.06	8.25	22.20	0.01	9.44	11,441.0	23,777.0	15,892.0
P20b	3	5	45.79	0.15	15.80	43.90	0.03	12.52	15,505.0	25,114.0	19,615.0
P20b	4	4	19.05	0.01	0.60	11.94	0.00	0.75	7092.0	18,972.0	12,160.0
P20b	4	5	10.16	0.02	1.06	8.24	0.00	0.89	8645.0	19,081.0	15,005.0
P20b	5	5	8.58	0.00	0.12	3.03	0.00	0.14	5408.0	15,055.0	11,038.0
P21a	3	3	39.16	0.02	10.16	39.36	0.00	8.74	4867.5	12,526.5	7462.5
P21a	3	4	61.08	0.09	18.02	46.97	0.01	17.49	6344.5	13,135.5	8405.5
P21a	3	5	159.26	0.22	36.53	259.45	0.03	30.10	8549.5	13,969.5	10,482.5
P21a	4	4	22.49	0.01	0.79	14.03	0.00	1.03	3921.5	10,431.5	7729.5
P21a	4	5	20.48	0.04	0.78	11.65	0.00	0.72	4732.5	10,405.5	8181.5
P21a	5	5	15.06	0.00	0.06	6.81	0.00	0.07	3032.5	8563.5	6076.5
P21b	3	3	63.26	0.02	4.30	35.04	0.00	6.92	8139.0	21,966.0	12,388.0
P21b	3	4	44.31	0.08	7.14	34.48	0.01	6.06	9809.0	22,333.0	13,575.0
P21b	3	5	166.81	0.25	23.08	132.33	0.02	24.52	12,928.0	22,681.0	16,204.0
P21b	4	4	24.52	0.01	0.54	16.35	0.00	0.84	6282.0	17,987.0	11,948.0
P21b	4	5	28.86	0.04	0.17	13.54	0.00	0.08	7931.0	17,942.0	13,682.0
P21b	5	5	5.37	0.00	0.25	3.32	0.00	0.12	5030.0	14,957.0	10,483.0
P21c	3	3	37.98	0.02	9.14	31.50	0.00	10.72	7846.0	20,665.0	12,427.0
P21c	3	4	149.09	0.10	35.04	103.93	0.01	38.77	8961.0	21,437.0	13,227.0

Table 3 continued

Name	m	i	Full1	Full 2	Full3	Cuts1	Cuts 2	Cuts3	opt 1	opt 2	opt 3
P21c	3	5	505.49	0.48	112.18	244.65	0.05	75.40	11,470.0	22,351.0	15,882.0
P21c	4	4	16.82	0.01	0.99	10.50	0.00	1.16	6804.0	17,283.0	12,625.0
P21c	4	5	31.26	0.04	1.04	16.82	0.00	0.82	7141.0	17,053.0	13,211.0
P21c	5	5	8.97	0.00	0.08	3.46	0.00	0.13	5323.0	14,221.0	10,377.0
P21d	3	3	42.58	0.02	7.52	34.02	0.00	7.38	8260.0	21,814.0	12,886.0
P21d	3	4	72.35	0.09	21.91	41.30	0.00	19.12	10,683.0	22,677.0	14,756.0
P21d	3	5	503.34	0.26	57.67	184.50	0.04	30.48	15,094.0	24,338.0	19,221.0
P21d	4	4	33.38	0.01	0.56	34.37	0.00	0.67	6940.0	18,225.0	12,828.0
P21d	4	5	13.02	0.04	0.65	9.53	0.00	0.36	8888.0	18,817.0	14,877.0
P21d	5	5	8.03	0.00	0.13	4.22	0.00	0.10	5268.0	15,208.0	11,024.0
P21e	3	3	136.03	0.02	7.87	109.80	0.00	8.70	9156.5	23,443.5	13,624.5
P21e	3	4	67.89	0.11	15.70	48.20	0.01	18.29	11,574.5	25,059.5	15,808.5
P21e	3	5	522.27	0.49	123.05	237.85	0.15	96.25	15,436.5	26,552.5	19,670.5
P21e	4	4	48.42	0.01	0.68	59.63	0.00	0.70	7351.5	19,820.5	13,372.5
P21e	4	5	13.86	0.04	1.00	10.83	0.00	1.16	8970.5	20,592.5	15,645.5
P21e	5	5	17.96	0.00	0.16	7.49	0.00	0.13	5541.5	16,655.5	11,441.5

We apply our basic model (9)–(20) for both variants Full and Cuts for the three distance types $\{1, 2, 3\}$

Table 4 Running times in seconds and optimal values "opt" for kPROP instances from [40] with $n \in \{22, 23\}$ with $\lfloor \frac{n}{i} \rfloor$ departments in each of the first $m - 1$ rows for the three distance types

Name	m	i	Full1	Full 2	Full3	Cuts1	Cuts 2	Cuts3	opt 1	opt 2	opt 3
P22a	3	3	140.06	0.04	16.98	347.53	0.00	13.82	6213.0	15,797.0	9829.0
P22a	3	4	175.68	0.14	22.45	112.13	0.02	25.23	8928.0	16,823.0	11,308.0
P22a	3	5	621.41	0.60	86.66	354.00	0.16	68.63	11,634.0	18,002.0	14,082.0
P22a	4	4	36.15	0.02	1.18	27.63	0.00	1.56	5476.0	13,453.0	10,477.0
P22a	4	5	47.60	0.07	1.26	22.87	0.01	1.25	6870.0	13,629.0	11,136.0
P22a	5	5	14.03	0.01	0.06	5.90	0.00	0.05	4286.0	10,994.0	8438.0
P22b	3	3	175.84	0.03	31.82	365.78	0.01	32.43	10,398.0	26,847.0	16,698.0
P22b	3	4	109.08	0.16	40.82	113.28	0.02	41.82	13,030.0	28,196.0	18,219.0
P22b	3	5	303.24	0.47	42.88	298.78	0.05	37.21	18,470.0	30,211.0	23,371.0
P22b	4	4	46.20	0.02	0.54	28.60	0.00	0.33	8168.0	22,115.0	15,805.0
P22b	4	5	39.10	0.08	2.37	46.28	0.01	2.58	10,509.0	23,566.0	18,285.0
P22b	5	5	19.04	0.01	0.16	8.14	0.00	0.14	6615.0	19,201.0	14,224.0
P22c	3	3	227.66	0.03	15.81	300.94	0.00	13.70	10,194.0	25,805.0	16,217.0
P22c	3	4	129.43	0.15	38.77	168.32	0.02	27.70	15,139.0	27,997.0	19,293.0
P22c	3	5	245.65	0.67	22.80	167.74	0.12	9.56	20,267.0	29,877.0	24,351.0
P22c	4	4	79.22	0.02	0.66	34.60	0.00	0.79	8564.0	21,739.0	15,642.0
P22c	4	5	35.16	0.07	1.57	22.71	0.00	0.84	12,139.0	23,237.0	18,810.0
P22c	5	5	32.69	0.01	0.07	10.02	0.00	0.04	6673.0	18,255.0	13,325.0

Table 4 continued

Name	m	i	Full1	Full 2	Full3	Cuts1	Cuts 2	Cuts3	opt 1	opt 2	opt 3
P22d	3	3	125.77	0.03	8.32	126.81	0.00	11.62	11,053.0	28,052.0	17,142.0
P22d	3	4	175.14	0.13	20.18	107.81	0.02	20.97	13,503.0	29,154.0	18,148.0
P22d	3	5	295.45	0.28	16.51	221.81	0.04	6.00	18,950.0	30,516.0	23,411.0
P22d	4	4	39.02	0.07	0.78	38.65	0.00	1.15	8308.0	23,241.0	15,939.0
P22d	4	5	52.28	0.07	0.90	39.00	0.00	0.19	9990.0	23,305.0	17,778.0
P22d	5	5	29.78	0.01	0.24	19.38	0.00	0.08	6624.0	19,337.0	14,165.0
P22e	3	3	232.98	0.04	10.56	344.94	0.01	12.74	10,336.5	25,655.5	16,349.5
P22e	3	4	69.70	0.30	47.30	49.80	0.07	42.82	17,355.5	30,707.5	22,528.5
P22e	3	5	216.20	1.58	73.89	187.95	0.38	82.93	21,414.5	31,821.5	25,897.5
P22e	4	4	31.86	0.02	0.54	18.16	0.00	0.43	8801.5	22,152.5	15,680.5
P22e	4	5	25.79	0.10	2.50	20.94	0.01	2.22	12,024.5	23,554.5	19,220.5
P22e	5	5	12.39	0.01	0.12	4.42	0.00	0.06	7004.5	18,244.5	13,972.5
P23a	3	3	288.02	0.06	21.65	627.37	0.01	24.35	7321.0	18,212.0	11,377.0
P23a	3	4	261.30	0.24	98.03	220.24	0.07	85.10	10,654.0	19,247.0	13,364.0
P23a	3	5	899.24	1.62	169.68	418.71	0.47	140.79	13,675.0	20,533.0	16,281.0
P23a	4	4	76.30	0.02	3.82	39.87	0.00	4.33	6583.0	15,593.0	12,294.0
P23a	4	5	67.81	0.10	4.65	67.28	0.07	4.54	8310.0	15,823.0	13,054.0
P23a	5	5	45.16	0.02	0.23	13.13	0.00	0.16	5293.0	12,773.0	10,104.0

Table 4 continued

Name	m	i	Full1	Full 2	Full3	Cuts1	Cuts 2	Cuts3	opt 1	opt 2	opt 3
P23b	3	3	99.62	0.05	14.34	66.89	0.00	13.94	11,493.5	27,227.5	18,063.5
P23b	3	4	116.28	0.22	66.99	128.87	0.04	30.92	16,696.5	29,712.5	21,965.5
P23b	3	5	215.68	0.79	185.51	177.70	0.14	90.74	21,930.5	31,951.5	26,267.5
P23b	4	4	21.05	0.02	0.52	14.20	0.00	0.63	9645.5	23,236.5	17,241.5
P23b	4	5	39.80	0.10	1.98	30.40	0.01	1.35	14,308.5	24,984.5	21,016.5
P23b	5	5	44.78	0.02	0.17	15.06	0.00	0.11	8213.5	19,185.5	15,057.5
P23c	3	3	133.90	0.05	9.40	87.63	0.00	8.88	11,293.0	26,913.0	17,061.0
P23c	3	4	137.73	0.26	38.75	72.25	0.04	39.66	17,701.0	29,983.0	22,247.0
P23c	3	5	271.91	0.81	147.51	203.32	0.13	113.30	22,115.0	31,484.0	26,304.0
P23c	4	4	38.76	0.02	0.62	29.77	0.00	0.60	9533.0	22,710.0	16,584.0
P23c	4	5	28.68	0.09	2.95	33.50	0.01	3.78	13,675.0	24,735.0	20,354.0
P23c	5	5	16.49	0.02	0.07	10.31	0.00	0.04	8001.0	19,280.0	14,882.0
P23d	3	3	429.18	0.05	28.36	2538.00	0.01	24.02	11,908.0	29,764.0	18,976.0
P23d	3	4	205.67	0.34	156.77	271.24	0.07	65.64	19,822.0	33,930.0	25,325.0
P23d	3	5	417.24	1.50	137.76	371.62	0.21	98.34	25,008.0	35,905.0	29,756.0
P23d	4	4	130.12	0.02	1.03	105.25	0.00	1.18	10,152.0	25,210.0	18,232.0
P23d	4	5	151.22	0.11	5.90	112.14	0.02	4.15	14,529.0	27,499.0	22,196.0
P23d	5	5	29.56	0.01	0.33	17.95	0.00	0.18	8267.0	21,500.0	16,596.0
P23e	3	3	168.72	0.04	17.77	247.80	0.01	20.58	11,199.5	29,222.5	17,644.5
P23e	3	4	298.84	0.22	45.91	163.06	0.02	33.37	16,011.5	30,390.5	20,315.5
P23e	3	5	502.23	0.56	126.94	246.18	0.15	46.46	20,218.5	32,014.5	23,977.5
P23e	4	4	97.55	0.02	1.88	88.00	0.00	1.75	9299.5	24,129.5	17,153.5
P23e	4	5	48.02	0.09	2.43	22.51	0.02	1.41	12,694.5	25,861.5	20,616.5
P23e	5	5	74.51	0.02	0.08	23.52	0.00	0.05	7778.5	20,840.5	15,339.5

We apply our basic model (9)–(20) for both variants Full and Cuts for the three distance types $\{1, 2, 3\}$

Table 5 Running times in seconds and optimal values "opt" for new kPROP instances (constructed according to the rules in [40]) with $n \in \{24, 25\}$ with $\lfloor \frac{n}{i} \rfloor$ departments in each of the first $m - 1$ rows for the three distance types

Name	m	i	Full1	Full 2	Full3	Cuts1	Cuts 2	Cuts3	opt 1	opt 2	opt 3
P24a	3	3	118.01	0.04	28.50	255.46	0.00	20.03	7786.0	20,415.0	12,529.0
P24a	3	4	253.38	0.20	57.04	238.65	0.04	43.51	10,206.0	21,008.0	14,529.0
P24a	3	5	818.97	2.55	354.76	776.10	0.75	183.97	16,196.0	23,426.0	19,037.0
P24a	4	4	165.09	0.02	1.34	258.16	0.00	2.68	6188.0	16,634.0	11,974.0
P24a	4	5	109.86	0.19	5.84	166.32	0.02	2.91	10,515.0	18,875.0	15,578.0
P24a	5	5	48.88	0.02	0.27	19.84	0.00	0.14	6828.0	15,278.0	12,441.0
P24b	3	3	334.46	0.04	16.62	693.66	0.00	12.33	12,181.5	30,661.5	19,127.5
P24b	3	4	775.58	0.20	36.91	236.16	0.03	46.09	15,535.5	32,002.5	22,680.5
P24b	3	5	927.38	3.47	180.90	1047.90	0.55	129.85	23,361.5	35,162.5	28,885.5
P24b	4	4	892.92	0.02	2.69	935.92	0.00	3.50	9775.5	24,864.5	18,587.5
P24b	4	5	127.09	0.18	9.38	110.72	0.03	5.52	15,551.5	28,839.5	24,020.5
P24b	5	5	143.02	0.02	0.70	45.95	0.00	0.66	10,142.5	22,914.5	18,711.5
P24c	3	3	278.67	0.04	72.54	537.79	0.00	26.84	11,724.5	29,736.5	18,531.5
P24c	3	4	1741.17	0.18	49.45	233.10	0.03	34.33	12,823.5	29,341.5	19,445.5
P24c	3	5	1273.55	3.69	154.14	809.34	0.42	82.49	20,319.5	32,661.5	25,373.5
P24c	4	4	89.03	0.02	3.47	54.06	0.00	4.24	9014.5	23,968.5	17,645.5
P24c	4	5	178.96	0.18	5.34	73.18	0.04	3.58	12,304.5	25,963.5	20,728.5
P24c	5	5	78.67	0.02	0.79	37.68	0.00	0.74	8335.5	21,249.5	17,221.5

Table 5 continued

Name	m	i	Full1	Full 2	Full3	Cuts1	Cuts 2	Cuts3	opt 1	opt 2	opt 3
P24d	3	3	112.90	0.04	5.38	126.19	0.00	7.76	13,153.0	34,340.0	20,817.0
P24d	3	4	202.85	0.18	30.76	117.84	0.03	19.30	16,013.0	34,898.0	24,223.0
P24d	3	5	1473.26	0.98	121.08	431.27	0.24	39.92	23,580.0	37,270.0	29,879.0
P24d	4	4	87.85	0.02	2.48	79.28	0.00	3.19	10,643.0	28,053.0	20,682.0
P24d	4	5	131.79	0.15	3.75	99.39	0.03	5.12	15,915.0	30,837.0	25,182.0
P24d	5	5	31.94	0.02	0.48	23.89	0.00	0.30	10,210.0	25,017.0	20,133.0
P24e	3	3	484.83	0.04	12.54	305.85	0.00	8.62	14,847.0	37,580.0	22,442.0
P24e	3	4	223.32	0.18	27.94	417.92	0.01	21.53	18,347.0	37,504.0	26,053.0
P24e	3	5	960.88	2.57	251.85	694.41	0.80	130.08	30,005.0	44,191.0	35,715.0
P24e	4	4	206.76	0.02	1.80	155.42	0.00	1.85	11,586.0	29,778.0	21,738.0
P24e	4	5	79.08	0.14	3.05	92.60	0.01	2.17	18,411.0	34,610.0	28,030.0
P24e	5	5	68.20	0.02	0.15	25.94	0.00	0.06	11,108.0	27,583.0	21,450.0
P25a	3	3	705.19	0.08	27.33	1282.28	0.02	20.16	8780.5	22,323.5	13,974.5
P25a	3	4	453.55	0.33	180.23	358.22	0.06	81.61	11,537.5	23,033.5	16,080.5
P25a	3	5	629.95	0.40	174.20	764.44	0.11	143.65	14,441.5	24,268.5	17,641.5
P25a	4	4	287.65	0.02	2.54	189.36	0.00	2.99	6796.5	18,182.5	13,266.5
P25a	4	5	395.02	0.10	18.10	202.24	0.02	14.69	9548.5	20,384.5	16,460.5
P25a	5	5	180.20	0.01	0.57	338.06	0.00	0.29	5768.5	15,860.5	11,994.5
P25b	3	3	4464.30	0.08	16.65	3312.89	0.01	13.44	12,129.0	30,894.0	18,268.0
P25b	3	4	544.10	0.37	60.41	277.46	0.07	60.24	16,150.0	31,616.0	22,842.0
P25b	3	5	831.44	1.19	171.65	523.27	0.29	116.37	20,351.0	33,785.0	25,577.0
P25b	4	4	320.23	0.03	2.32	520.19	0.00	2.51	9249.0	24,873.0	18,005.0
P25b	4	5	1774.64	0.07	6.36	541.94	0.01	3.90	11,535.0	26,776.0	19,962.0
P25b	5	5	781.15	0.01	0.45	805.00	0.00	0.83	7729.0	21,576.0	15,786.0

Table 5 continued

Name	m	i	Full1	Full 2	Full3	Cuts1	Cuts 2	Cuts3	opt 1	opt 2	opt 3
P25c	3	3	3988.06	0.08	43.67	4023.36	0.01	41.41	16,183.5	41,087.5	25,683.5
P25c	3	4	348.07	0.48	158.04	335.25	0.09	78.08	21,955.5	42,947.5	29,963.5
P25c	3	5	648.34	1.31	244.32	714.66	0.28	116.18	28,287.5	45,689.5	34,432.5
P25c	4	4	372.03	0.02	2.85	2714.81	0.00	4.27	12,655.5	32,883.5	24,086.5
P25c	4	5	581.06	0.08	2.78	206.31	0.01	3.49	17,132.5	35,907.5	27,800.5
P25c	5	5	494.43	0.01	0.24	180.37	0.00	0.36	9801.5	28,173.5	20,661.5
P25d	3	3	583.23	0.07	37.54	1632.38	0.01	29.38	15,379.0	39,071.0	23,844.0
P25d	3	4	437.83	0.33	40.48	979.16	0.04	67.55	19,769.0	39,277.0	27,603.0
P25d	3	5	794.45	0.92	319.39	469.68	0.26	218.32	24,073.0	42,118.0	30,638.0
P25d	4	4	2080.20	0.02	3.56	1589.86	0.00	3.12	12,059.0	31,212.0	22,699.0
P25d	4	5	751.79	0.07	3.25	823.52	0.01	3.95	14,899.0	33,490.0	25,399.0
P25d	5	5	170.88	0.01	0.10	223.36	0.00	0.06	9761.0	27,381.0	19,716.0
P25e	3	3	759.49	0.06	31.86	1819.47	0.01	39.97	14,361.5	36,781.5	22,521.5
P25e	3	4	561.74	0.38	98.12	959.77	0.08	64.41	16,990.5	37,058.5	24,521.5
P25e	3	5	902.66	0.70	307.10	725.57	0.14	238.42	20,420.5	38,502.5	26,683.5
P25e	4	4	193.72	0.02	2.41	366.81	0.00	3.34	11,278.5	30,009.5	21,618.5
P25e	4	5	338.44	0.08	4.74	238.20	0.02	4.30	13,310.5	31,756.5	23,752.5
P25e	5	5	159.88	0.01	0.60	38.76	0.00	1.16	9329.5	26,121.5	19,098.5

We apply our basic model (9)–(20) for both variants Full and Cuts for the three distance types $\{1, 2, 3\}$

6.2 Results for the SF-DRFLP and the DRFLP

Iterating over all possible row assignments and neglecting all unbalanced assignments, we can solve the SF-DRFLP and the DRFLP. Although the number of row assignments grows exponentially and so lots of NP-hard problems had to be solved, we could solve instances with up to 16 departments in reasonable time, see Tables 6 and 7. Note, the previously largest instance of the SF-DRFLP contained 13 departments and the largest DRFLP instances contained 15 departments [69]. To allow the reader a direct comparison to previous approaches from the literature we also tested the models in [9] and their extensions according to [68,69] (for details we refer to the Appendix) for SF-DRFLP and the models in [10,68,69] for DRFLP and included the running times as well as the gaps, calculated by $\frac{\text{optimal}}{\text{lower bound}} - 1$, for instances not solved within the time limit of 12 h in Tables 6 and 7. With our new approach all instances with $n \leq 15$ could be solved in less than 2 h, only the instances with $n = 16$ were costlier, but could be solved within the time limit. This was not possible for five SF-DRFLP instances and for nine DRFLP instances using the approaches of Amaral [9,10] as well as for four SF-DRFLP and four DRFLP instances using the approaches of Secchin and Amaral [69]. Usually our approach is much faster. But note that the approaches from the literature were faster than our new ones for very small instances. For SF-DRFLP the approach in [9] is often better than the extended one in [69] for instances with $n \leq 13$, but for larger instances the running times or the gaps can be improved by including betweenness-type variables. A similar behavior can be observed for DRFLP. Comparing Full and Cuts, Cuts was better in almost all cases for DRFLP. The situation is not so clear for SF-DRFLP, where Full is several times better, but the differences are often not very high. So we advice the reader to use cutting planes.

In order to show the impact of the investigations in Sect. 5 we also tested our enumeration scheme for the SF-DRFLP without using Lemma 6, i.e., we considered all assignments with $L_1 \geq L_2$, neglecting the ones with $L_1 = L_2$ and $r_1 = 2$. In our newly presented variant we additionally restricted to those assignments with $\ell_{\max,1} \geq L_1 - L_2$. For the DRFLP we tested $IP_{\texttt{FR-DRFLP}}$ with the big-M-value $M = \sum_{i=1}^{n} \ell_i$ and without excluding row assignments where the sum of the lengths of the departments in one of the rows exceeds t defined in (39). All results are included in Table 6 for the SF-DRFLP and Table 7 for the DRFLP. These tables show that reducing the number of assignments that have to be considered allowed to reduce the running times of the SF-DRFLP significantly. For the DRFLP the combined effect of the improved big-M-value and the exclusion of unbalanced row assignments was much smaller for our model, here only small improvements were possible. One reason for this behavior might be that inequalities (36) and (37) bound the distances between the departments rather well from below. Additionally, we could only exclude few row assignments based on the maximal layout length t. The effect was stronger for the approach of Amaral. In most cases the running times or the gaps could be improved significantly using the better big-M-value and the exclusion of unbalanced row assignments. For [69] the improved big-M-value slows down the solution approach several times. One reason is again that it is harder to detect good upper bounds. But there are several instances as well where the new big-M-value helped, see especially instance

Table 6 Running times in seconds and optimal values "Optimal" for the `SF-DRFLP` obtained by applying our MILP models for both variants `Full` and `Cuts` for each relevant row assignment in the standard variant ($L_1 \geq L_2$) and according to Lemma 6

Name	Source	Standard		Lemma 6		Amaral	Secchin-A.	Optimal
		Full	Cuts	Full	Cuts	time (gap)	time (gap)	
Am11a	[3]	107.68	109.29	74.27	89.11	322.42	347.18	5631.5
Am11b	[3]	48.93	37.86	23.43	23.13	408.34	456.38	3664.5
Am11c	[3]	62.03	52.36	35.31	39.01	430.06	511.62	3832.5
Am11d	[3]	61.81	54.54	37.55	42.48	141.42	230.19	906.5
Am11e	[3]	34.08	19.15	10.34	9.77	65.06	154.70	584.0
Am11f	[3]	52.53	46.04	25.60	35.70	131.69	281.17	845.0
Am12a	[9,10]	177.97	145.79	111.78	117.92	598.60	564.42	1529.0
Am12b	[9,10]	120.97	84.16	74.58	57.84	721.13	740.11	1609.5
Am12c	[3]	130.24	91.20	67.19	66.19	635.26	743.52	2043.0
Am12d	[3]	87.36	50.00	28.21	29.08	301.70	416.91	1116.0
Am12e	[3]	94.23	52.54	30.32	28.77	405.34	667.26	1073.0
Am12f	[3]	117.41	78.98	49.64	51.98	420.93	535.73	998.0
Am13a	[9]	461.86	343.34	287.97	287.18	2721.84	3078.69	2467.5
Am13b	[9]	400.59	267.65	226.00	208.12	2048.54	1470.52	2870.0
Am13c	[3]	491.46	387.70	308.00	323.90	2557.17	2923.29	4155.0
Am13d	[3]	311.54	178.52	129.44	110.26	3979.29	5120.89	6164.5
Am13e	[3]	372.10	252.27	180.44	164.37	4645.74	5686.46	6511.5
Am13f	[3]	702.39	648.18	388.24	422.88	7325.88	9639.05	7718.5
Am14_1	New	2004.23	1681.63	1418.14	1450.48	15,633.12	14,578.01	2756.5

Table 6 continued

Name	Source	Standard		Lemma 6		Amaral	Secchin-A.	Optimal
		Full	Cuts	Full	Cuts	time (gap)	time (gap)	
Am14a	[69]	1830.01	1490.19	1358.83	1317.93	25,182.37	15,357.48	2921.0
Am14b	[69]	1029.35	661.32	534.74	455.89	20,509.60	16,362.76	2748.0
Am15_1	[4]	3064.81	1750.66	1697.45	1290.15	TL (0.10)	TL (0.03)	3195.0
HA5	[42]	0.03	0.01	0.01	0.00	0.01	0.02	52.5
HA6	[42]	0.15	0.14	0.10	0.12	0.05	0.08	190.5
HA7	[42]	0.32	0.29	0.24	0.27	0.09	0.24	166.0
HA8	[42]	1.52	1.39	1.18	1.01	0.62	1.20	205.0
HA9	[42]	6.46	7.14	3.86	5.09	6.08	5.70	492.5
HA10	[42]	20.08	18.78	12.11	14.90	30.24	54.74	838.0
HA11	[42]	32.52	22.12	15.11	16.53	43.52	108.93	796.0
HA12	[42]	193.84	169.10	126.76	155.97	462.46	609.33	1028.0
HA13	[42]	334.08	233.58	167.00	166.42	1113.72	1492.24	1530.5
HA14	[42]	1990.72	1459.98	1225.04	1071.54	16,757.58	15,714.74	1841.0
HA15	[42]	8007.61	7117.44	5616.50	5910.14	TL (0.06)	TL (0.03)	2643.5
HK15	[38]	2527.16	1164.32	1266.68	782.39	TL (0.01)	28,080.32	16,640.0
P16_a	[11]	39,809.57	37,818.40	29,431.25	29,836.62	TL (0.71)	TL (0.60)	7370.0
P16_b	[11]	13,538.96	8916.02	8550.54	6671.26	* (0.51)	TL (0.49)	5884.5
Small.10-1	[3]	9.76	6.08	2.98	3.12	27.46	44.59	1389.5
Small.10-2	[3]	10.65	6.69	3.52	3.52	33.42	57.32	1437.5
Small.10-3	[3]	11.58	8.08	4.29	4.48	63.39	66.02	1461.0
Small.10-4	[3]	11.08	6.74	4.00	3.84	56.76	51.88	1326.0

Table 6 continued

Name	Source	Standard		Lemma 6		Amaral	Secchin-A.	Optimal
		Full	Cuts	Full	Cuts	time (gap)	time (gap)	
Small.10-5	[3]	11.33	7.26	4.43	4.15	25.09	52.90	736.5
Small.10-6	[3]	11.53	8.16	4.73	5.14	26.70	49.31	803.5
Small.10-7	[3]	12.27	9.35	8.56	5.74	30.83	51.86	622.5
Small.10-8	[3]	9.95	5.83	5.74	2.82	16.39	36.90	540.0
Small.10-9	[3]	13.02	9.71	4.60	5.28	45.74	53.23	933.5
Small.10-10	[3]	19.15	18.08	10.02	14.80	40.84	66.80	859.5
s9	[9,10]	5.42	3.03	1.64	1.83	11.44	7.54	1181.5
s9h	[9,10]	20.35	29.11	12.28	19.11	77.39	50.88	2294.5
s10	[9,10]	12.44	8.56	4.48	6.15	38.22	63.33	1374.5
s11	[9,10]	52.03	43.50	23.72	25.34	205.94	229.00	3439.5

Additionally, the table shows the running times and the gaps after a time limit TL of 12 h in brackets using the model of Amaral [9] and of Secchin and Amaral [69]. The symbol “∗” indicates that the computer ran out of memory

Table 7 Running times in seconds and optimal values "Optimal" for the `DRFLP` obtained by applying our MILP models for both variants `Full` and `Cuts` for each relevant row assignment with big-M-value as given

Name	Source	$M = \sum_{i=1}^{n} \ell_i$				$M = \sum_{i=\lfloor \frac{n+1}{3} \rfloor +1}^{n} \ell_i$				Optimal
		Full	Cuts	Amaral	Secchin-A.	Full	Cuts	Amaral	Secchin-A.	
Am11a	[3]	67.46	39.11	685.92	327.81	60.73	38.37	528.36	254.00	5559.0
Am11b	[3]	66.38	21.52	910.45	348.53	35.47	17.62	856.94	300.71	3655.5
Am11c	[3]	63.14	43.39	676.03	416.27	53.61	38.74	620.29	674.82	3832.5
Am11d	[3]	59.69	42.37	278.30	172.15	60.10	42.60	322.58	150.29	906.5
Am11e	[3]	31.75	12.03	160.62	105.18	26.51	11.19	81.07	106.16	578.0
Am11f	[3]	39.70	23.47	140.91	104.96	35.44	18.83	84.59	97.04	825.5
Am12a	[9,10]	123.80	49.80	791.72	493.11	116.84	46.83	840.47	488.30	1493.0
Am12b	[9,10]	118.23	41.07	727.88	503.35	129.08	41.23	1010.62	456.81	1606.5
Am12c	[3]	114.13	35.03	1052.32	467.44	104.02	33.16	1425.34	730.31	2012.5
Am12d	[3]	93.44	24.68	180.61	229.62	78.45	23.22	270.30	241.36	1107.0
Am12e	[3]	97.48	28.44	450.87	403.39	80.22	25.97	410.41	345.46	1066.0
Am12f	[3]	120.78	52.93	805.05	468.88	112.32	51.66	538.39	420.78	997.5
Am13a	[9]	467.50	224.64	6813.18	1941.73	458.96	227.94	5697.01	3116.92	2456.5
Am13b	[9]	426.98	166.04	6232.85	1858.45	408.67	161.45	4291.77	1919.54	2864.0
Am13c	[3]	474.31	225.80	4885.02	2820.36	466.07	223.42	3375.96	1736.32	4136.0
Am13d	[3]	357.76	107.58	7070.02	4710.27	334.08	104.84	4983.34	3714.11	6164.5
Am13e	[3]	387.86	143.30	21,887.97	4599.66	375.72	145.04	16,361.44	5285.90	6502.5
Am13f	[3]	642.87	411.06	14,422.20	6995.22	625.89	410.47	16,022.18	5809.93	7699.5
Am14_1	new	1821.60	856.60	TL (0.07)	21,548.33	1756.87	847.90	TL (0.05)	23,171.06	2738.5

Table 7 continued

Name	Source	$M = \sum_{i=1}^{n} \ell_i$				$M = \sum_{i=\lfloor \frac{n+1}{3} \rfloor +1}^{n} \ell_i$				Optimal
		Full	Cuts	Amaral	Secchin-A.	Full	Cuts	Amaral	Secchin-A.	
Am14a	[69]	1642.64	788.31	TL (0.14)	24,152.30	1604.38	777.84	TL (0.04)	29,081.03	2904.0
Am14b	[69]	1109.40	323.39	TL (0.16)	32,620.24	1055.12	312.43	TL (0.02)	34,365.43	2736.0
Am15_1	[4]	3980.13	1232.89	TL (0.34)	TL (0.12)	3899.50	1212.28	TL (0.36)	TL (0.12)	3195.0
HA5	[42]	0.02	0.01	0.00	0.01	0.01	0.00	0.00	0.02	52.5
HA6	[42]	0.10	0.08	0.06	0.06	0.08	0.07	0.05	0.06	190.5
HA7	[42]	0.31	0.22	0.10	0.14	0.26	0.24	0.08	0.14	159.0
HA8	[42]	0.94	0.42	1.42	0.91	0.69	0.40	2.96	1.02	189.5
HA9	[42]	4.16	3.07	6.96	4.06	4.25	2.86	7.70	4.17	486.5
HA10	[42]	16.90	12.62	40.75	27.02	16.01	12.65	33.25	29.56	821.0
HA11	[42]	31.18	9.46	38.87	50.09	27.16	8.56	46.08	48.44	773.5
HA12	[42]	183.51	115.60	856.68	372.57	175.55	115.25	707.06	541.08	1021.0
HA13	[42]	386.19	149.95	3478.34	1790.69	446.17	144.10	2425.04	1389.72	1520.5
HA14	[42]	1914.47	929.88	TL (0.08)	24,663.75	1843.48	915.08	TL (0.33)	20,792.79	1833.5
HA15	[42]	6625.08	3865.76	TL (0.19)	TL (0.08)	6559.96	3786.67	TL (0.22)	42,805.24	2624.5
HK15	[38]	3109.96	623.00	TL (0.14)	24,634.80	3046.99	583.93	TL (0.07)	TL (0.06)	16,570.0
P16_a	[11]	36,032.00	28,016.60	TL (1.00)	TL (0.69)	35,100.67	27,614.73	TL (0.86)	TL (0.74)	7365.5
P16_b	[11]	15,690.86	6338.54	TL (0.97)	TL (0.62)	15,445.70	6277.02	* (0.87)	TL (0.60)	5870.5
Small.10-1	[3]	10.24	2.99	41.70	40.39	9.16	2.82	36.81	47.27	1385.0
Small.10-2	[3]	10.84	3.89	40.30	48.83	9.80	3.64	34.83	48.98	1437.0
Small.10-3	[3]	10.28	2.85	47.54	39.03	9.06	2.61	41.64	44.48	1452.5
Small.10-4	[3]	10.28	3.04	37.53	33.55	9.19	3.07	31.06	35.78	1313.5
Small.10-5	[3]	10.17	3.16	21.65	29.46	8.78	2.84	19.02	25.78	722.5
Small.10-6	[3]	10.29	3.64	23.72	27.28	8.92	3.04	30.22	28.61	792.0

Table 7 continued

Name	Source	$M = \sum_{i=1}^{n} \ell_i$				$M = \sum_{i=\lfloor \frac{n+1}{3} \rfloor +1}^{n} \ell_i$				Optimal
		Full	Cuts	Amaral	Secchin-A.	Full	Cuts	Amaral	Secchin-A.	
Small.10-7	[3]	13.83	3.69	28.88	29.73	9.78	3.41	21.54	28.28	607.5
Small.10-8	[3]	16.68	2.60	36.20	15.66	7.97	2.53	11.75	16.76	529.0
Small.10-9	[3]	13.58	6.62	34.88	33.53	10.80	10.30	31.04	32.22	929.5
Small.10-10	[3]	15.53	5.12	33.78	34.23	10.21	9.55	37.71	35.84	828.0
s9	[9,10]	4.05	2.12	8.67	11.71	3.50	1.99	9.53	12.24	1179.0
s9h	[9,10]	19.54	32.70	133.51	54.57	16.06	33.59	140.03	55.48	2293.0
s10	[9,10]	12.45	3.35	41.40	37.71	9.89	3.24	46.88	37.36	1351.0
s11	[9,10]	47.82	20.47	467.72	306.61	35.60	16.87	370.80	338.62	3424.5

Additionally, the table shows the running times and the gaps after a time limit TL of 12 h in brackets using the model of Amaral [10] and Secchin and Amaral [69]. The symbol "$*$" indicates that the computer ran out of memory

Table 8 Running times in seconds and optimal values “Optimal” for the `SF-MRFLP` with $m \in \{3, 4, 5\}$ using our enumeration scheme using separation (`Cuts`) and the balancing of the two longest rows according to Corollary 8

Name	Source	Time			Optimal		
		$m = 3$	$m = 4$	$m = 5$	$m = 3$	$m = 4$	$m = 5$
Am11a	[3]	221.85	497.47	663.31	3763.5	2761.5	2156.5
Am11b	[3]	157.30	462.33	720.08	2403.5	1816.5	1446.5
Am11c	[3]	192.02	512.12	687.41	2504.5	1906.5	1438.5
Am11d	[3]	126.52	336.00	565.26	532.5	389.5	305.5
Am11e	[3]	188.00	368.72	613.11	416.0	298.0	254.0
Am11f	[3]	184.26	413.51	568.83	558.0	408.0	297.0
Am12a	[9,10]	648.28	2648.94	4341.10	1028.0	785.0	629.0
Am12b	[9,10]	798.60	2678.24	4707.03	1117.5	855.5	617.5
Am12c	[3]	522.02	2362.38	4173.54	1344.0	1009.0	806.0
Am12d	[3]	425.86	1567.27	3039.42	734.0	530.0	417.0
Am12e	[3]	462.27	1533.48	2875.37	701.0	505.0	391.0
Am12f	[3]	548.66	1726.60	3396.61	661.0	452.0	366.0
Am13a	[9]	2058.19	8909.18	19,346.42	1569.5	1145.5	882.5
Am13b	[9]	2631.50	10,129.68	21,744.91	1904.0	1372.0	1085.0
Am13c	[3]	2074.15	9738.88	20,597.51	2659.0	1933.0	1456.0
Am13d	[3]	2013.40	11,617.26	22,380.68	4069.5	3086.5	2361.5
Am13e	[3]	2605.13	13,190.14	26,320.47	4340.5	3233.5	2502.5
Am13f	[3]	2296.95	15,025.67	29,431.88	5011.5	3847.5	3095.5
HA5	[42]	0.02	0.02	0.02	34.5	34.5	34.5
HA6	[42]	0.16	0.20	0.20	110.5	98.5	94.5
HA7	[42]	0.38	0.50	0.50	108.0	77.0	61.0
HA8	[42]	2.24	2.82	3.51	138.0	98.0	82.0
HA9	[42]	10.46	13.05	17.74	317.5	212.5	157.5
HA10	[42]	49.50	72.10	101.90	541.0	363.0	272.0
HA11	[42]	157.98	360.90	522.87	547.0	394.0	294.0
HA12	[42]	832.92	2083.55	3482.10	674.0	483.0	376.0
HA13	[42]	2970.84	12,500.24	25,311.54	1019.5	750.5	611.5
Small.09-1	[3]	19.27	26.82	23.41	1532.0	1152.0	870.0
Small.09-2	[3]	14.91	21.90	22.29	1398.0	1052.0	823.0
Small.09-3	[3]	17.81	25.57	22.17	1389.0	1039.0	779.0
Small.09-4	[3]	19.00	22.30	22.65	1393.5	1017.5	781.5
Small.09-5	[3]	30.68	27.35	24.64	2457.5	1866.5	1399.5
Small.09-6	[3]	34.46	28.61	25.76	2553.0	1914.0	1441.0
Small.09-7	[3]	22.65	28.93	23.42	2529.0	1905.0	1404.0
Small.09-8	[3]	24.93	28.80	24.82	2587.5	1917.5	1457.5

Table 8 continued

Name	Source	Time			Optimal		
		$m = 3$	$m = 4$	$m = 5$	$m = 3$	$m = 4$	$m = 5$
Small.09-9	[3]	9.40	21.41	17.99	579.0	450.0	318.0
Small.09-10	[3]	18.39	18.84	20.47	691.5	485.5	362.5
Small.10-1	[3]	32.10	66.51	97.72	918.5	653.5	534.5
Small.10-2	[3]	35.51	66.01	102.63	968.5	666.5	560.5
Small.10-3	[3]	26.32	64.82	101.96	933.0	667.0	556.0
Small.10-4	[3]	23.97	59.69	94.67	834.0	596.0	497.0
Small.10-5	[3]	28.42	65.72	95.21	458.5	349.5	279.5
Small.10-6	[3]	34.00	69.32	100.38	519.5	374.5	307.5
Small.10-7	[3]	31.20	63.71	99.18	382.5	297.5	237.5
Small.10-8	[3]	21.36	48.38	80.83	328.0	245.0	197.0
Small.10-9	[3]	39.72	84.96	105.17	619.5	447.5	351.5
Small.10-10	[3]	35.00	83.28	107.90	547.5	413.5	331.5
s9	[9,10]	8.34	18.05	21.98	771.5	587.5	472.5
s9h	[9,10]	19.64	26.57	22.12	1431.5	1062.5	817.5
s10	[9,10]	27.89	64.03	98.07	887.5	630.5	499.5
s11	[9,10]	167.88	523.70	687.37	2308.5	1743.5	1350.5

HA15 which could not be solved to optimality within the time limit using the standard variant.

6.3 Results for the `SF-MRFLP` and the `MRFLP`

Our enumeration scheme allows us to solve `SF-MRFLP` and `MRFLP` instances as well. Because of the very large number of subproblems that have to be solved we use all possible row number reductions. By the results for the `DRFLP` and the `SF-DRFLP` we decided to test only variant `Cuts`. The results for instances with $n \leq 13$ can be found in Tables 8 and 9. One can see that the number of rows has a significant impact on the running times. So our approach is much faster for $m = 3$ for $n \geq 10$. Interestingly, for smaller instances the situation is not so clear. Then, partially we have the smallest running times for $m = 5$ for the `MRFLP`. One reason for this might be that the number of interesting row assignments is rather small if m is rather large in comparison to n. Comparing the `SF-MRFLP` and the `MRFLP` the running times are usually higher for the `SF-MRFLP`. One reason for this behavior might be the number of row assignments. For the `MRFLP` we can always assume that each row contains at least one department. As shown in Fig. 7 for the `SF-MRFLP` we can only assume that two rows are non-empty. Encouraged by the good running times for $m = 3$ we tested instances with $n \in \{14, 15\}$ in this case as well. The result are shown in Table 10. Again the `MRFLP` seems to be better to solve than the `SF-MRFLP`. All `MRFLP` instances could be solved within 12 h. For `SF-MRFLP` this was not possible for two instances with $n = 15$.

Table 9 Running times in seconds and optimal values “Optimal” for the MRFLP with $m \in \{3, 4, 5\}$ using our enumeration scheme using separation (Cuts) and our improved big-M-value (32)

Name	Source	Time			Optimal		
		$m = 3$	$m = 4$	$m = 5$	$m = 3$	$m = 4$	$m = 5$
Am11a	[3]	102.08	310.40	413.70	3688.5	2722.5	2101.0
Am11b	[3]	104.44	295.98	393.57	2385.5	1759.5	1356.0
Am11c	[3]	136.50	318.50	391.73	2504.5	1844.0	1335.0
Am11d	[3]	89.44	296.26	417.01	527.5	377.0	272.5
Am11e	[3]	122.68	386.06	534.05	392.0	286.5	220.5
Am11f	[3]	155.34	347.75	438.09	543.5	380.0	275.5
Am12a	[9,10]	366.06	1633.52	2890.90	1010.0	731.0	565.0
Am12b	[9,10]	386.14	1645.22	2915.16	1052.5	777.5	593.5
Am12c	[3]	320.92	1920.00	2792.52	1290.0	940.0	707.5
Am12d	[3]	540.83	1402.36	2620.83	708.0	477.0	389.5
Am12e	[3]	348.70	1373.74	3473.24	681.5	443.5	349.0
Am12f	[3]	409.08	1741.50	2866.87	633.0	440.0	339.0
Am13a	[9]	1220.68	7261.65	16,537.70	1551.0	1130.0	836.0
Am13b	[9]	1584.37	8313.00	18,103.11	1835.0	1312.0	991.5
Am13c	[3]	1450.36	8147.14	17,468.23	2614.5	1872.0	1384.0
Am13d	[3]	1430.65	7802.50	17,304.96	4035.5	2928.0	2236.5
Am13e	[3]	1795.80	8758.89	18,704.82	4312.5	3158.5	2360.5
Am13f	[3]	1733.56	9531.18	18,577.81	5011.5	3797.5	2970.5
HA5	[42]	0.02	0.00	0.00	18.0	10.0	0.0
HA6	[42]	0.08	0.04	0.01	96.0	55.0	24.0
HA7	[42]	0.38	0.35	0.12	93.0	59.5	38.5
HA8	[42]	1.31	1.80	1.19	118.5	90.0	71.0
HA9	[42]	6.74	10.88	8.88	298.5	194.5	130.0
HA10	[42]	28.62	57.83	58.16	509.0	348.5	237.5
HA11	[42]	97.24	329.15	440.71	521.0	371.5	282.0
HA12	[42]	665.62	2250.84	2947.71	659.0	468.0	343.0
HA13	[42]	2023.95	10,223.73	22,584.46	983.5	711.5	538.5
Small.09-1	[3]	23.72	23.02	9.66	1516.0	1138.0	848.0
Small.09-2	[3]	20.63	15.39	9.32	1398.0	1039.0	802.5
Small.09-3	[3]	23.78	25.52	10.28	1381.5	1034.0	771.5
Small.09-4	[3]	27.22	19.70	10.45	1388.0	1014.0	781.5
Small.09-5	[3]	37.67	24.52	10.70	2457.5	1847.5	1372.5
Small.09-6	[3]	42.15	28.78	10.79	2546.0	1900.0	1416.0
Small.09-7	[3]	28.46	27.27	10.18	2500.5	1885.5	1368.5
Small.09-8	[3]	32.82	24.92	9.72	2558.5	1899.5	1402.5
Small.09-9	[3]	9.79	15.51	8.37	575.5	448.5	307.5
Small.09-10	[3]	22.81	14.54	8.00	686.0	485.0	352.5
Small.10-1	[3]	21.18	52.74	56.82	906.5	609.0	447.5

Table 9 continued

Name	Source	Time			Optimal		
		$m=3$	$m=4$	$m=5$	$m=3$	$m=4$	$m=5$
Small.10-2	[3]	21.71	53.22	56.99	948.0	617.5	466.0
Small.10-3	[3]	20.40	52.96	57.12	928.5	635.0	483.0
Small.10-4	[3]	19.99	52.44	56.94	826.5	563.5	430.5
Small.10-5	[3]	19.14	51.91	58.18	439.5	318.0	234.0
Small.10-6	[3]	20.24	51.96	56.86	487.5	338.5	254.0
Small.10-7	[3]	20.64	55.96	61.02	365.5	268.5	203.5
Small.10-8	[3]	17.77	49.54	56.14	303.5	214.5	156.5
Small.10-9	[3]	24.09	56.56	59.24	577.0	411.0	302.0
Small.10-10	[3]	22.12	55.74	60.25	511.5	361.0	270.0
s9	[9,10]	6.28	10.56	8.26	757.0	543.5	429.5
s9h	[9,10]	24.05	21.06	9.14	1413.5	1048.5	786.5
s10	[9,10]	20.84	52.44	56.83	868.0	578.5	427.5
s11	[9,10]	96.38	310.04	386.30	2263.5	1689.5	1230.0

Table 10 Running times in seconds and optimal values for SF-MRFLP and MRFLP with $n \in \{14, 15\}$ and $m = 3$ using our enumeration scheme using separation (Cuts), Corollary 8 for SF-MRFLP and our improved big-M-value (32) for MRFLP

Name	Source	Time		Optimal	
		SF-MRFLP	MRFLP	SF-MRFLP	MRFLP
Am14_1	New	27,585.46	9204.24	1911.5	1858.5
Am14a	[69]	14,660.87	8354.82	2006.0	1991.0
Am14b	[69]	23,224.93	8383.20	1885.0	1842.0
Am15_1	[4]	50,046.74∗	29,612.48	2167.0	2163.0
HA14	[42]	10,156.10	8805.50	1242.0	1225.0
HA15	[42]	62,758.55∗	39,161.08	1741.5	1724.0
HK15	[38]	29,238.42	19,502.98	10,880.0	10,720.0

Note that the running times of two SF-MRFLP instances marked with "∗" exceeded 12 h

7 Conclusion and future work

In this paper we presented a new formulation for the kPROP. Combining this formulation and a slightly modified model allowing spaces with an enumeration scheme iterating over all relevant row assignments we were able to solve SF-DRFLP and DRFLP instances with up to 16 departments for the first time and to reduce the running times for smaller instances significantly. Apart from this we solved SF-MRFLP and MRFLP instances with up to 15 departments for three rows and with up to 13 departments for four and five rows. To further speed-up the enumeration scheme we proved with the help of combinatorial arguments that very unbalanced row assignments do not have to be considered in the enumeration in both cases, i.e., with and without allowing space between neighboring departments.

It remains for future work to improve this approach. One direction could be the study of the corresponding polyhedra deriving stronger relaxations for the kPROP and the FR-MRFLP. Furthermore, assigning most but not all of the departments to the rows, a large amount of the total objective value is predetermined. It would be nice if one could detect situations that cannot lead to optimal solutions beforehand, e.g., by calculating some combinatorial bounds.

In our enumeration scheme we did not use, the partially known, upper bounds on the optimal objective value, determined by some heuristic. We only compared to the currently best solution that was found with our enumeration scheme without employing additional improvement heuristics. So it would also be worth to study the impact of using heuristics and if one can find some criteria which row assignments should be considered first. At the moment we use the same order of the row assignments for all instances.

Our approach is mainly based on the combination of a fast solution of kPROP and FR-MRFLP instances and an enumeration scheme. In general, one could try to combine the assignment of the departments to the rows and the inter-row distance calculation via betweenness variables and further variables, coupled with appropriate constraints, in a single model. For this note that most MILP models for the DRFLP in the literature do not use betweenness variables, although these were successfully employed for the SRFLP.

As a next step it would also be interesting to consider facility layout problems with more complex path structures, if, e.g., there are two paths in the shape of a T or an X.

Acknowledgements This work was supported by the Simulation Science Center Clausthal–Göttingen. We thank two anonymous referees for their valuable comments that helped to improve the paper. Furthermore we thank A. Amaral for sending us several DRFLP instances.

Appendix

In the following we repeat the DRFLP model of Secchin and Amaral presented in [68, 69] and show how these ideas can be used to improve the SF-DRFLP model of Amaral [9] as well. For an DRFLP instance given by n departments with lengths $\ell_i, i \in [n]$, pairwise transport weights $w_{ij}, i, j \in [n], i < j$, and two rows we use the position variables $\tilde{x}_i, i \in [n]$, distance variables $\tilde{d}_{ij}, i, j \in [n], i < j$, and binary ordering variables $\tilde{\alpha}_{ij}, i, j \in [n], i \neq j$, that are one if and only if department i lies left to department j in the same row, otherwise they are zero. In comparison to [10] we additionally use the binary betweenness-type variables $\tilde{y}_{ijk}, i, j, k \in [n], |\{i, j, k\}| = 3, i < k$, with the interpretation

$$\tilde{y}_{ijk} = \begin{cases} 1, & j \text{ lies between departments } i, k \text{ in the same row,} \\ 0, & \text{otherwise.} \end{cases}$$

We set $L := \sum_{i\in[n]} \ell_i$. Then the model reads:

$$
\begin{aligned}
\min \; & \sum_{\substack{i,j\in[n]\\ i<j}} w_{ij}\tilde{d}_{ij} && \\
\text{s.t. } & \tilde{d}_{ij} \geq \tilde{x}_i - \tilde{x}_j, && i,j\in[n], i<j, \\
& \tilde{d}_{ij} \geq \tilde{x}_j - \tilde{x}_i, && i,j\in[n], i<j, \\
& \tilde{x}_{i^*} \leq \tilde{x}_{j^*}, && \text{for some fixed } (i^*,j^*) \in \\
& && \operatorname*{Argmin}_{i,j\in[n],i<j} \{w_{ij}\}, \\
& -\tilde{\alpha}_{ij} + \tilde{\alpha}_{ik} + \tilde{\alpha}_{jk} - \tilde{\alpha}_{ji} + \tilde{\alpha}_{ki} + \tilde{\alpha}_{kj} \leq 1, && i,j,k\in[n], |\{i,j,k\}| = 3, \\
& && i<j, \qquad (40) \\
& \tilde{\alpha}_{ij} + \tilde{\alpha}_{ik} - \tilde{\alpha}_{jk} + \tilde{\alpha}_{ji} - \tilde{\alpha}_{ki} + \tilde{\alpha}_{kj} \leq 1, && i,j,k\in[n], i\neq k, \\
& && i<j, k<j, \qquad (41) \\
& \tilde{\alpha}_{ij} + \tilde{\alpha}_{ik} + \tilde{\alpha}_{jk} + \tilde{\alpha}_{ji} + \tilde{\alpha}_{ki} + \tilde{\alpha}_{kj} \geq 1, && i,j,k\in[n], i<j<k, \qquad (42) \\
& \tilde{x}_i + \tilde{d}_{ij} \leq \tilde{x}_j + 2(L - \tfrac{\ell_i}{2} - \tfrac{\ell_j}{2})(1-\tilde{\alpha}_{ij}), && i,j\in[n], i<j, \\
& \tilde{x}_i + \tilde{d}_{ji} \leq \tilde{x}_j + 2(L - \tfrac{\ell_i}{2} - \tfrac{\ell_j}{2})(1-\tilde{\alpha}_{ij}), && i,j\in[n], j<i, \\
& \tilde{y}_{ikj} \geq \tilde{\alpha}_{ik} + \tilde{\alpha}_{kj} - 1, && i,j,k\in[n], |\{i,j,k\}| = 3, \\
& && i<j, \qquad (43) \\
& \tilde{y}_{ikj} \geq \tilde{\alpha}_{jk} + \tilde{\alpha}_{ki} - 1, && i,j,k\in[n], |\{i,j,k\}| = 3, \\
& && i<j, \qquad (44) \\
& \tilde{d}_{ij} \geq (\tfrac{\ell_i+\ell_j}{2})(\tilde{\alpha}_{ij} + \tilde{\alpha}_{ji}) + \sum_{\substack{k\in[n]\\ k\neq i,j}} \ell_k \tilde{y}_{ikj}, && i,j\in[n], i<j, \qquad (45) \\
& \tfrac{\ell_i}{2} \leq \tilde{x}_i \leq L - \tfrac{\ell_i}{2}, && i\in[n], \\
& \tilde{\alpha}_{ij} \in \{0,1\}, && i,j\in[n], i\neq j, \qquad (46) \\
& \tilde{y}_{ijk} \in \{0,1\}, && i,j,k\in[n], |\{i,j,k\}| = 3, \\
& && i<k. \qquad (47)
\end{aligned}
$$

Additionally, Secchin and Amaral introduced the following constraints:

$$
\left.\begin{aligned}
\tilde{d}_{ij} &\leq \tilde{d}_{ik} + \tilde{d}_{jk}, \\
\tilde{d}_{ik} &\leq \tilde{d}_{ij} + \tilde{d}_{jk}, \\
\tilde{d}_{jk} &\leq \tilde{d}_{ij} + \tilde{d}_{ik},
\end{aligned}\right\} \qquad i,j,k\in[n], i<j<k, \tag{48}
$$

$$
\left.\begin{aligned}
\tilde{y}_{ikj} &\leq \tilde{\alpha}_{ik} + \tilde{\alpha}_{ki}, \\
\tilde{y}_{ikj} &\leq \tilde{\alpha}_{ij} + \tilde{\alpha}_{ji}, \\
\tilde{y}_{ikj} &\leq \tilde{\alpha}_{jk} + \tilde{\alpha}_{kj},
\end{aligned}\right\} \qquad i,j,k\in[n], |\{i,j,k\}| = 3, i<j. \tag{49}
$$

After an analysis of the running times of different separation variants, it was suggested in [69] to use the triangle constraints (48) from the beginning and to not use (49). In our tests in Sect. 6, where we compare our new models with approaches from the literature, we use the same variant. Additionally, for the DRFLP we strengthened the model above by reducing the value of L according to Lemma 10.

Finally, we show how the SF-DRFLP model in [9] can be strengthened using the ideas above. In this model position variables $\tilde{x}_i$, $i \in [n]$, are not needed because the left border is fixed and spaces between neighboring departments are not allowed. So one can get rid of them. Using the variables $\tilde{\alpha}_{ij}, \tilde{d}_{ij}, i, j \in [n], i \neq j$, as well as the betweenness variables $\tilde{y}_{ijk}, i, j, k \in [n], |\{i, j, k\}| = 3, i < k$, from above the extended model reads:

$$
\begin{aligned}
\min \quad & \sum_{\substack{i,j \in [n] \\ i<j}} w_{ij} \tilde{d}_{ij} \\
\text{s.t.} \quad & (40) - (47) \\
& \tilde{d}_{ij} \geq \tfrac{\ell_i - \ell_j}{2} + \sum_{k \in [n] \setminus \{i\}} \ell_k \tilde{\alpha}_{ki} - \sum_{k \in [n] \setminus \{j\}} \ell_k \tilde{\alpha}_{kj}, \qquad i, j \in [n], i < j, \\
& \tilde{d}_{ij} \geq \tfrac{\ell_j - \ell_i}{2} + \sum_{k \in [n] \setminus \{j\}} \ell_k \tilde{\alpha}_{kj} - \sum_{k \in [n] \setminus \{i\}} \ell_k \tilde{\alpha}_{ki}, \qquad i, j \in [n], i < j.
\end{aligned}
$$

This model can be improved by using (48) and (49). Additionally, one could fix $\tilde{\alpha}_{\hat{\imath}\hat{\jmath}}$ to zero for one pair $\hat{\imath}, \hat{\jmath} \in [n], \hat{\imath} < \hat{\jmath}$. In our computational tests we added (48) from the beginning and did not use constraints (49) as suggested for the DRFLP in [69].

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