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FULL LENGTH PAPER

Intersection cuts for single row corner relaxations

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Abstract We consider the problem of generating inequalities that are valid for one-row relaxations of a simplex tableau, with the integrality constraints preserved for one or more non-basic variables. These relaxations are interesting because they can be used to generate cutting planes for general mixed-integer problems. We first consider the case of a single non-basic integer variable. This relaxation is related to a simple knapsack set with two integer variables and two continuous variables. We study its facial structure by rewriting it as a constrained two-row model, and prove that all its facets arise from a finite number of maximal $(\mathbb{Z} \times \mathbb{Z}_+)$-free splits and wedges. The resulting cuts generalize both MIR and 2-step MIR inequalities. Then, we describe an algorithm for enumerating all the maximal $(\mathbb{Z} \times \mathbb{Z}_+)$-free sets corresponding to facet-defining inequalities, and we provide an upper bound on the split rank of those inequalities. Finally, we run computational experiments to compare the strength of wedge cuts against MIR cuts. In our computations, we use the so-called trivial fill-in function to exploit the integrality of more non-basic variables. To that end, we present a practical algorithm for computing the coefficients of this lifting function.

Keywords Integer programming · Lifting · Cutting planes

Mathematics Subject Classification 90C11 · 90C57

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1 Introduction

Since the early days of integer programming, cut-generating functions [34] and intersection cuts [6] have provided a theoretical foundation for computing a wide range of valid inequalities for the convex hull of the integer feasible region. Yet, the most important classes of general-purpose cutting planes used in practice, such as Gomory mixed-integer cuts [31] and mixed-integer rounding inequalities [42,43], are generated either from a single row, or from a single linear combination of rows of the simplex tableau. However, a 2007 paper from Andersen et al. [4] triggered a renewed interest in the study of inequalities that can only be generated when considering two or more tableau rows simultaneously. More specifically, some form of the following mixed-integer model was studied in [4,6,8,11,13,14,19,23,27,32,39]:

$$\begin{aligned} & x = f + \sum_{j \in N} r^j s_j, \\ & x \in S, \\ & s_j \in \mathbb{R}_+, \ \text{for all } j \in N \end{aligned} \tag{1}$$

where $f \in \mathbb{Q}^m \setminus \mathbb{Z}^m$, $r^j \in \mathbb{Q}^m$ for $j \in N$, and S is the set of integral points contained in some rational polyhedron in $\mathbb{R}^m$. The usual approach to obtain this model from a general MIP is to consider some simplex tableau of its LP relaxation, then drop the rows in which the basic variable is continuous and relax integrality constraints on non-basic variables. The latter constraints, however, can be exploited by adopting a *lifting* approach [10,15,22,24,25]: First, the integral non-basic variables are fixed to zero. This amounts to removing the corresponding columns from the problem, yielding a model of the form (1). A facet-defining inequality is generated for this model. Then, the missing variables are re-introduced, and corresponding valid coefficients are computed, while the coefficients of the continuous variables are kept unchanged. In other words, an initial inequality $\alpha^T s \geq 1$ is *lifted* into a higher-dimensional space, yielding an inequality $\gamma^T y + \alpha^T s \geq 1$ that is valid for

$$\begin{aligned} & x = f + \sum_{j \in K} r^j y_j + \sum_{j \in N} r^j s_j, \\ & x \in S, \\ & y_j \in \mathbb{Z}_+, \ \text{for all } j \in K, \\ & s_j \in \mathbb{R}_+, \ \text{for all } j \in N. \end{aligned} \tag{2}$$

Given α, a lifting γ is said to be *minimal* if there does not exist a valid inequality ${\gamma'}^T y + \alpha^T s \geq 1$ for (2) that is distinct from $\gamma^T y + \alpha^T s \geq 1$ and dominates it. In other words, it is minimal if there is no valid inequality ${\gamma'}^T y + \alpha^T s \geq 1$ such that $\gamma' \neq \gamma$ and $\gamma'_j \leq \gamma_j$ for all j. Furthermore, the lifting is *unique* (or *sequence-independent*) if there does not exist a valid inequality ${\gamma''}^T y + \alpha^T s \geq 1$ for (2) that is distinct from $\gamma^T y + \alpha^T s \geq 1$ and minimal.

Note that with this approach, even in the simplest case where a unique minimal lifting exists and can be computed, not all facet-defining inequalities for (2) can be obtained; only those inequalities for which the α coefficients form a facet-defining inequality for (1). Instead, we are interested in characterizing more facet-defining inequalities of (2); in particular, inequalities that cannot be obtained through lifting of facet-defining inequalities for the continuous model (1). In this paper, we focus on the single-row case ($m = 1$) with $S = \mathbb{Z}$.

We start by considering the special case where there is a single integral non-basic variable ($|K| = 1$). We assume for the sake of conciseness that we have continuous variables with both positive and negative coefficients. The model can then be simplified by aggregating them according to the sign of their coefficient. That is, we study the structure of the set

$$P = \left\{ (x, s) \in \mathbb{Z} \times \mathbb{R}^3_+ \ : \ x = \phi + \rho s_1 + s_2 - s_3, \ s_1 \in \mathbb{Z} \right\} \tag{3}$$

for some $\phi \in \mathbb{Q} \setminus \mathbb{Z}$ and $\rho \in \mathbb{Q}$. By considering s_3 as the slack of an inequality constraint, we can see that P is closely related to the set of solutions of a mixed-integer knapsack problem having two integral variables and one continuous variable. Hirschberg and Wong [37] developed a polynomial-time algorithm to optimize over pure integer knapsack problems with two variables. Agra and Constantino [1,2] provided a complete characterization of $\text{conv}(P)$, and a polynomial-time method exploiting the approach in [37] to enumerate its facet-defining inequalities. Similar results are also due to Atamtürk and Rajan [5]. The particularity of our approach is that we use the framework of multi-row intersection cuts [6]. By doing so, we obtain a nice geometric interpretation of our results. In particular, this yields a natural upper bound on the split rank of the integer hull of P. Moreover, the tools we develop are particularly well-suited for a practical implementation, and we present computational results using our cuts on MIPLIB 2010 [38] instances.

Despite our multi-row approach, we are presenting tools to evaluate the computational strength of a subset of one-row cuts, more commonly known as *knapsack* cuts, and there is insightful previous work on this topic. Fischetti and Saturni [26] developed a separation procedure for interpolated subadditive cuts, which are another subset of knapsack cuts and are valid for the master cyclic group polyhedron characterized by Gomory and Johnson [32–34]. These cuts include Gomory's mixed-integer (GMI) cuts [31], which are the reference in this context since (i) only one GMI cut can be computed per knapsack (setting aside integer scaling), (ii) GMI cuts can be computed easily via a simple formula, and (iii) they are known to be useful computationally in the context of branch-and-bound [9]. But from their results, Fischetti and Saturni suggest that on MIPLIB benchmark instances, the cuts they separate offer little gain beyond that of GMI cuts. Dash and Günlük [20] reached similar conclusions considering all group cuts. They even observe that for a significant fraction of the problem instances, GMIs were the only relevant group cuts. Fukasawa and Goycoolea [28] further generalized these approaches by separating over all knapsack cuts, but still show that the whole class of knapsack cuts together outperforms GMI cuts only by a small margin. Even going to multiple rows (or knapsacks), Louveaux et al. [40] show that the gap with GMI cuts widens for 2-, 3- and 5-row cuts, but still not in an overly promising way given the computational costs involved. This sets the context in terms of computational expectations for our cuts (which are group cuts). However, we are aiming at generating our cuts at a much lower computational cost than, e.g., [20,26,28,40].

In Sect. 2, we rewrite P as a two-row model, and show that all the facet-defining inequalities for $\text{conv}(P)$ are intersection cuts obtained either from a split unbounded along the line $\binom{\phi}{0} + \lambda \binom{\rho}{1}$, or from a finite number of wedges whose vertex lies on the same line. In Sect. 3, we present an algorithm to enumerate all the sets that yield facet-defining intersection cuts. We exploit this algorithm in Sect. 4 to compute an upper bound on the split rank of the corresponding facets.

Next, we tackle the problem of exploiting integrality constraints on more non-basic variables. Our approach is more traditional in this case. We compute the cut coefficients of the additional integral variables by making use of the trivial lifting [7,33] (or trivial fill-in [22,24]) function. In Sect. 5, we present a practical algorithm to compute these coefficients. Finally, in Sect.6, we run computational experiments to compare the strength of the cuts developed here against MIR cuts [43] (which they generalize) alone. While, for most problems, wedge cuts did not improve the integrality gap significantly, for some instances they did present a clear improvement, under very reasonable running times.

2 The case of a single integral non-basic variable

In this section, we start by considering a one-row model where the integrality of a single non-basic variable is preserved. More precisely, we study the structure of the set P in (3). As suggested by Conforti et al. [15], this set can be rewritten as

$$P_I = \left\{(x, s) \in S \times \mathbb{R}^3_+ : \binom{x_1}{x_2} = \binom{\phi}{0} + \binom{\rho}{1} s_1 + \binom{1}{0} s_2 + \binom{-1}{0} s_3\right\},$$

where we let $S := (\mathbb{Z} \times \mathbb{Z}_+)$. Note that we use $S = (\mathbb{Z} \times \mathbb{Z}_+)$ to emphasize that x_2 is nonnegative, although $S = \mathbb{Z}^2$ would yield the same set since $x_2 = s_1$ and $s_1 \geq 0$. We now have a two-row model P_I for which all s variables are continuous. Let $f = \binom{\phi}{0}$, $r^1 := \binom{\rho}{1}$, $r^2 := \binom{1}{0}$, $r^3 := \binom{-1}{0}$ and $R := \left[r^1|r^2|r^3\right]$, i.e.,

$$P_I = \left\{(x, s) \in S \times \mathbb{R}^3_+ : x = f + Rs\right\}.$$

We will use f and $\begin{pmatrix}\phi\\0\end{pmatrix}$ (resp. r^1 and $\binom{\rho}{1}$) interchangeably throughout this section, depending on which notation better conveys the intuition. Our definition of $\text{conv}(P_I)$ is a special case of the set with the same name in [4], and the following properties carry over from [4]:

Proposition 1 [4] (i) *The dimension of* $\text{conv}(P_I)$ *is three.*
(ii) *The extreme rays of* $\text{conv}(P_I)$ *are* $(\rho, 1, 1, 0, 0)$, $(1, 0, 0, 1, 0)$ *and* $(-1, 0, 0, 0, 1)$.

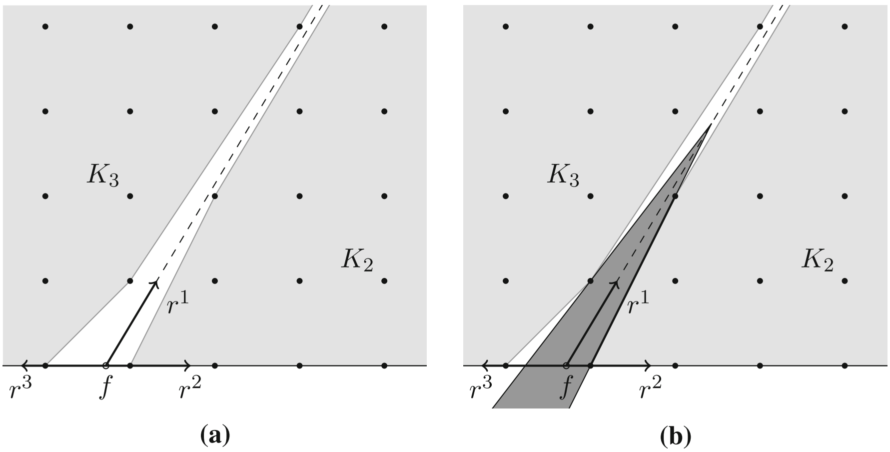

Fig. 1 Knapsack sets and facet-defining S-free sets. **a** Knapsack sets K_2 and K_3. **b** Example of facet-defining wedge

Closely related to the structure of $\text{conv}(P_I)$ are the two knapsack sets

$$K_j = \mathbb{Z}^2 \cap \Big(f + \text{cone}(r^1, r^j)\Big) \quad \text{for } j \in \{2, 3\},$$

illustrated in Fig. 1a. For the vertices of $\text{conv}(P_I)$, we can refine the characterization from [4]. Note that we use e_j to denote the jth column of the 3×3 identity matrix, and $\text{lin}(r)$ as the linear subspace generated by a vector $r \in \mathbb{R}^2$, i.e., $\text{lin}(r) := \{\lambda r : \lambda \in \mathbb{R}\}$.

Proposition 2 *A point $(\bar{x}, \bar{s}) \in P_I$ is a vertex of* $\text{conv}(P_I)$ *if and only if $\bar{s} = \bar{s}_1 e_1 + \bar{s}_j e_j$ for some $j \in \{2, 3\}$ and $\bar{x}$ is a vertex of* $\text{conv}(K_j)$.

Proof ($\Rightarrow$) Assume that $(\bar{x}, \bar{s})$ is a vertex of $\text{conv}(P_I)$. Then, $\bar{x}$ is integer and $\bar{s}$ is a vertex of $P_I \cap \{(x, s) : x = \bar{x}\}$, hence a basic feasible solution to the system $\{s \in \mathbb{R}^3_+ : Rs = \bar{x} - f\}$. Thus, $\bar{s}$ has at most two nonzero components. Furthermore, since the submatrix $[r^2|r^3]$ is not invertible, either s_2 or s_3 is non-basic, hence zero. Therefore, $\bar{s} = \bar{s}_1 e_1 + \bar{s}_j e_j$ for some $j \in \{2, 3\}$. Since $\bar{x}$ is integer, this implies that $\bar{x} \in K_j$. We next show that $\bar{x}$ is a vertex of $\text{conv}(K_j)$. In the following, we assume $j = 2$, since the other case is similar. Suppose, by contradiction, that $\bar{x}$ is not a vertex of $\text{conv}(K_2)$. Then, there must exist $x^1, \ldots, x^k \in K_2 \cap \mathbb{Z}^2$ distinct from $\bar{x}$ and $\lambda \in \mathbb{R}^k_+$ such that $\bar{x} = \sum_{i=1}^k \lambda_i x^i$ and $\sum_{i=1}^k \lambda_i = 1$. Let $M = [r^1|r^2]$. Note that since $\text{lin}(r^1) \neq \text{lin}(r^2)$, M is invertible. For each $i \in \{1, \ldots, k\}$, let $s^i \in \mathbb{R}^3_+$ be such that $s^i = s^i_1 e_1 + s^i_2 e_2$ and

$$\begin{pmatrix} s^i_1 \\ s^i_2 \end{pmatrix} = M^{-1}(x^i - f).$$

For every $i \in \{1, \ldots, k\}$, $s_1^i, s_2^i \geq 0$ because $x^i \in K_2$, so $(x^i, s^i) \in P_I$. Furthermore, by linearity, $\bar{s} = \sum_{i=1}^k \lambda_i s^i$, thus $(\bar{x}, \bar{s}) = \sum_{i=1}^k \lambda_i (x^i, s^i)$. This contradicts the assumption that $(\bar{x}, \bar{s})$ is a vertex of $\text{conv}(P_I)$.

($\Leftarrow$) Let $(\bar{x}, \bar{s}) \in P_I$ be such that $\bar{s} = \bar{s}_1 e_1 + \bar{s}_j e_j$ for some $j \in \{2, 3\}$ and $\bar{x}$ is a vertex of $\text{conv}(K_j)$. In the following, we assume $j = 2$, since the other case is similar. We prove that $(\bar{x}, \bar{s})$ is a vertex of $\text{conv}(P_I)$. Suppose, by contradiction, that this is not the case. Then, there must exist k points $(x^1, s^1), \ldots, (x^k, s^k) \in P_I$ distinct from $(\bar{x}, \bar{s})$ and $\lambda \in \mathbb{R}_+^k$ such that $(\bar{x}, \bar{s}) = \sum_{i=1}^k \lambda_i (x^i, s^i)$ and $\sum_{i=1}^k \lambda_i = 1$. Since $\bar{s}_3 = 0$ and $\lambda \geq 0$, we have $s_3^i = 0$ for all i. Therefore $x^i \in K_2$ for all i and these points are all distinct from $\bar{x}$. We can construct $\bar{x}$ as a convex combination of k points $x^1, \ldots, x^k \in K_2$ distinct from $\bar{x}$. This contradicts the assumption that $\bar{x}$ is a vertex of $\text{conv}(K_2)$. □

We now look at the facet-defining inequalities for $\text{conv}(P_I)$.

Proposition 3 [4] *The facet-defining inequalities of* $\text{conv}(P_I)$ *take the form*

(i) $s_j \geq 0$ *for* $j \in \{1, 2, 3\}$,
(ii) $\alpha^T s \geq 1$ *for some* $\alpha \geq 0$.

Note that inequalities of the form (i) in Proposition 3, i.e., $s_j \geq 0$ for some $j \in \{1, 2, 3\}$, are called *trivial*, while those of the form (ii) are called *nontrivial*. For the nontrivial inequalities, we have the following further characterization.

Proposition 4 *Every nontrivial facet-defining inequality* $\alpha^T s \geq 1$ *of* $\text{conv}(P_I)$ *satisfies* $\alpha_2 > 0$ *and* $\alpha_3 > 0$. *If* $\alpha_1 = 0$, *then there are no integer points on the ray* $f + \text{cone}(r^1)$, *and there is only one facet-defining inequality of that form.*

Proof Let $z^2 := (\lceil\phi\rceil, 0, 0, \lceil\phi\rceil - \phi, 0)$ and $z^3 := (\lfloor\phi\rfloor, 0, 0, 0, \phi - \lfloor\phi\rfloor)$. Since z^2 and z^3 belong to P_I, we must have $\alpha_2 > 0$ and $\alpha_3 > 0$, respectively. Suppose $f + \lambda r^1 = \bar{x} \in \mathbb{Z}^2$ for some $\lambda \in \mathbb{R}_+$. Since $\phi \notin \mathbb{Z}$ we have $\lambda > 0$. Then $(\bar{x}_1, \bar{x}_2, \lambda, 0, 0) \in P_I$, and therefore $\alpha_1 > 0$. It follows that if $\alpha_1 = 0$, then $f + \lambda r^1 = \bar{x} \in \mathbb{Z}^2$ does not exist. Finally, we show uniqueness for a facet-defining inequality with $\alpha_1 = 0$. Suppose that $\alpha_2 s_2 + \alpha_3 s_3 \geq 1$ and $\alpha_2' s_2 + \alpha_3' s_3 \geq 1$ are facet-defining for $\text{conv}(P_I)$. Consider the vertices of $\text{conv}(P_I)$ that are tight on $\alpha_2 s_2 + \alpha_3 s_3 \geq 1$. By Proposition 2, they all have $s_h = 0$ for some $h \in \{2, 3\}$. However, the value of h is not the same for all of them, otherwise we could set $\alpha_h = 0$ and the resulting inequality would cut off z^h. Let $(\bar{x}, \bar{s})$ be one such vertex and let $\{j\} := \{2, 3\} \setminus \{h\}$. Since $\alpha_2' s_2 + \alpha_3' s_3 \geq 1$ is valid, $\alpha_j' \geq \alpha_j$. By applying the process to all vertices, then repeating for those that are tight on $\alpha_2' s_2 + \alpha_3' s_3 \geq 1$, we obtain $\alpha_2' = \alpha_2$ and $\alpha_3' = \alpha_3$. □

Our motivation for studying a model of the form of P_I is that such model is an ideal setting for computing and using intersection cuts [6]. Specifically, every nontrivial valid inequality for P_I is an intersection cut from some S-free set in $\mathbb{R}^2$ [23]. A convex set $B \subseteq \mathbb{R}^m$ is *S-free* if its interior contains f but no point of S. The set is *maximal* if it is not properly contained in any other S-free set. Maximal sets are the only ones that interest us, since any non-dominated inequality can be obtained from such sets. Note that every maximal S-free set is polyhedral [11,41], and given a polyhedral S-free set $B := \left\{x \in \mathbb{R}^m \ : \ g_i^T(x - f) \leq 1, \ i = 1, \ldots, k\right\}$, the intersection cut coefficient for s_j is given by $\psi_B(r^j) = \max_{i=1,\ldots,k} g_i^T r^j$ [23]. In the context of $\text{conv}(P_I)$, $x \in S = \mathbb{Z} \times \mathbb{Z}_+$ and $s \in \mathbb{R}^3_+$. Proposition 5 shows that in this case, we may restrict our attention to S-free sets B with two faces, i.e., $k = 2$. An analogous result was obtained in [15] for an infinite relaxation of P_I.

Proposition 5 *If $\alpha^T s \geq 1$ is a nontrivial valid inequality for P_I, then there exists an S-free set*

$$B = \left\{x \in \mathbb{R}^2 \ : \ g_1^T\,(x - f) \leq 1, \ g_2^T\,(x - f) \leq 1\right\}$$

such that $\alpha^T s \geq 1$ is the intersection cut computed from B.

Proposition 5 has a very simple justification: Only the intersections (if any) of the facets of B with the line $\text{lin}(r^j)$ affect the intersection cut coefficient α_j. Therefore, for a given cut $\alpha \in \mathbb{R}^3_+$, and one can easily construct a wedge or a split in $\mathbb{R}^2$ that provides the three desired intersections. It implies that all facet-defining inequalities for $\text{conv}(P_I)$ can be obtained from maximal S-free splits unbounded along the line $f + \text{lin}(r^1)$ and maximal S-free *wedges* whose vertex lies on that same line. As this reasoning relies on a geometric intuition for intersection cuts, we also provide a formal proof. Following [15], we use the term *wedge* to describe sets of the form $a + \text{cone}(b, c)$ with $a, b, c \in \mathbb{R}^2$, as illustrated in Fig. 1b.

Proof of Proposition 5 The proof is constructive. Let $\alpha^T s \geq 1$ be a nontrivial valid inequality for P_I. By Proposition 3, $\alpha \geq 0$, and by Proposition 4, $\alpha_2, \alpha_3 > 0$. We let $g_1 := (\alpha_2, \ \alpha_1 - \rho\alpha_2)$ and $g_2 := (-\alpha_3, \ \alpha_1 + \rho\alpha_3)$. It is straightforward to verify that B then yields the appropriate intersection cut coefficients. Suppose that B is not S-free. Then, there exists $\bar{x} \in S$ such that $g_1^T\,(\bar{x} - f) < 1$ and $g_2^T\,(\bar{x} - f) < 1$. We construct $\bar{s}$ such that $(\bar{x}, \bar{s}) \in P_I$. By substituting $\bar{x} - f = Rs$ in the two above inequalities, we obtain $\alpha_1\bar{s}_1 + \alpha_2\bar{s}_2 - \alpha_2\bar{s}_3 < 1$ and $\alpha_1\bar{s}_1 - \alpha_3\bar{s}_2 + \alpha_3\bar{s}_3 < 1$, respectively. We can assume without loss of generality that either $\bar{s}_2 = 0$ or $\bar{s}_3 = 0$. In each case, one of the latter inequalities yields $\alpha^T\bar{s} < 1$, which contradicts the validity of $\alpha^T s \geq 1$ for P_I. □

An interesting feature of the set B constructed above is that a vertex $(\bar{x}, \bar{s})$ of P_I is tight on $\alpha^T s \geq 1$ if and only if $\bar{x}$ is on the boundary of B. Indeed, the latter implies either $g_1^T\,(x - f) = 1$ (if $\bar{s}_3 = 0$), or $g_2^T\,(x - f) = 1$ (if $\bar{s}_2 = 0$). Again, substituting $x - f = Rs$ yields $\alpha^T\bar{s} = 1$ in both cases.

We now prove that we can restrict our attention even further, to a specific finite family of splits and wedges. This will let us develop an algorithm to enumerate all these relevant S-free sets in Sect. 3. Proposition 4 states that if $\alpha^T s \geq 1$ is facet-defining for $\text{conv}(P_I)$, then $\alpha_2, \alpha_3 > 0$. If $\alpha_1 = 0$, then there is exactly one facet-defining inequality of that form. The proof of Proposition 5 gives us the set $B = \{x \in \mathbb{R}^2 :$ $\frac{1}{\alpha_2} \leq \binom{-1}{\rho}^T\,(x - f) \leq \frac{1}{\alpha_3}\}$, called a *split* set, where α_2 and α_3 can be computed in a straightforward manner[1]. Otherwise, $\alpha > 0$ and B is a wedge with its apex on the line $f + \text{lin}(r^1)$. Then, Theorem 6 gives a useful characterization of the corresponding facet-defining inequalities.

Theorem 6 (i) *A valid inequality* $\alpha^T s \geq 1$ *where* $\alpha > 0$ *is facet-defining for* $\text{conv}(P_I)$ *if and only if it is tight at three distinct vertices of* $\text{conv}(P_I)$. (ii) *Furthermore, at least one of those three vertices corresponds to a vertex of* $\text{conv}(K_2)$*, and at least one corresponds to a vertex of* $\text{conv}(K_3)$.

Proof Let $P_s := \text{proj}_s \, \text{conv}(P_I)$ be the projection of $\text{conv}(P_I)$ on the space of the s variables. (i) $\Leftarrow$: Since $\dim(P_s) = 3$, a valid inequality that is tight at three affinely independent points is facet-defining. (i) $\Rightarrow$: Since $\dim(P_s) = 3$, a facet of P_s may contain fewer than three vertices of P_s only if its affine hull contains an extreme ray of P_s, specifically e_j for some $j \in \{1, 2, 3\}$, by Proposition 1. Assume that $\alpha^T s \geq 1$ is a corresponding facet-defining inequality that is tight at $\bar{s} \in P_s$, i.e., $\alpha^T \bar{s} = 1$. Then, $\alpha^T(\bar{s} + e_j) = 1$, implying that $\alpha_j = 0$. This contradicts $\alpha > 0$. (ii): Assume that three tight vertices (x^1, s^1), (x^2, s^2), (x^3, s^3) of $\text{conv}(P_I)$ correspond to three vertices x^1, x^2, x^3 of $\text{conv}(K_j)$, for a single fixed $j \in \{2, 3\}$. Let $\{h\} = \{2, 3\} \setminus \{j\}$. Then, $s_h^1 = s_h^2 = s_h^3 = 0$. The facet-defining inequality of $\text{conv}(P_I)$ that is tight at these three vertices is $s_h \geq 0$ (Proposition 3), contradicting $\alpha > 0$. □

Theorem 6 means that in order to obtain facet-defining intersection cuts for P_I, one should focus on S-free sets that have at least three S points on their boundary: at least one of each of K_2 and K_3. This means that each of those S-free sets is tight at two points of either K_2 or K_3. In other words, one of its facets coincides with a facet of either $\text{conv}(K_2)$ or $\text{conv}(K_3)$. See Fig. 1b. An analogous result is well-known in the case of an infinite relaxation of P_I [15,23].

3 Enumerating the vertices of the knapsacks

In this section we describe a simple algorithm for enumerating the vertices of the two knapsack sets K_2 and K_3 described in Sect. 2, allowing us to enumerate all the splits and wedges that induce facets of $\text{conv}(P_I)$.

Since we have a complete description of the extreme points and rays of $\text{conv}(P_I)$, its facet-defining inequalities could be obtained by enumerating the vertices of its polar, as shown by Andersen et al. [3,4] in dimension two, and Basu et al. [13] in general dimensions. Although this approach has been performed [39], it has two drawbacks: Even separation in two dimensions relies on optimizing over a cut-generating linear program (CGLP) with the simplex method, which adds a source of numerical inaccuracies. Then, finding *all* facet-defining inequalities would require enumerating the vertices of this CGLP, a difficult computational task. Here, instead, we exploit the

[1] If f and ρ are rational numbers, we can compute geometrically a maximal lattice-free set of that form. Specifically, letting $d \in \mathbb{Z}$ such that $fd \in \mathbb{Z}$ and $\rho d \in \mathbb{Z}$, $g = \gcd(d, \rho d)$ and $v = \frac{fd}{g} - \left\lfloor \frac{fd}{g} \right\rfloor$, we get the cut $\frac{g}{d(1-v)} s_2 + \frac{g}{dv} s_3 \geq 1$, provided that $\frac{fd}{g} \notin \mathbb{Z}$.

characterization provided by Theorem 6 to enumerate the facet-defining inequalities of $\text{conv}(P_I)$.

Enumerating the vertices of the knapsack sets K_2 and K_3 is a particular case of the integer hull problem. Harvey [36] devised an algorithm for enumerating the facets of the integer hull of an arbitrary two-dimensional polyhedron. The complexity of the algorithm is $O(n \log A_{\max})$ where n is the number of input inequalities and $A_{\max}$ is the magnitude of the largest input coefficient. This algorithm is optimal in the sense that no better asymptotic bound is possible for the problem. In the more specific case of a two-dimensional knapsack set, Agra and Constantino [1,2] and Atamtürk and Rajan [5] independently gave polynomial-time algorithms. Both are based on the two-dimensional knapsack optimization algorithm of Hirschberg and Wong [37].

Despite the abundant earlier work on the topic, we develop a different method for computing the vertices of the integer hull of a knapsack, with the following motivation. First, our method has a simple geometric interpretation that allows us to prove an upper bound on the split rank of $\text{conv}(P_I)$ (Sect. 4). Secondly, while also being optimal in the above sense, it is easy to implement and yields a very fast code, which we use in our computations (Sect. 6).

Throughout this section, for any $z \in \mathbb{R}$, we denote by $\widehat{z}$ the fractional part of z, i.e., $\widehat{z} := z - \lfloor z \rfloor$.

Consider the two sets

$$A = \text{conv}\left(\mathbb{Z}^2 \cap \left(\left(\begin{smallmatrix}\phi\\0\end{smallmatrix}\right) + \text{cone}\{\left(\begin{smallmatrix}\rho\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}-1\\0\end{smallmatrix}\right)\}\right)\right),$$
$$B = \text{conv}\left(\mathbb{Z}^2 \cap \left(\left(\begin{smallmatrix}\phi\\0\end{smallmatrix}\right) + \text{cone}\{\left(\begin{smallmatrix}\rho\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\}\right)\right).$$

Observe that A is simply $\text{conv}(K_3)$ and B is $\text{conv}(K_2)$. Our goal is to obtain the set of vertices of A and of B. For simplicity, we assume $0 < \phi < 1$. If that is not the case, A and B can be translated along the x_1 axis to enforce the assumption; the resulting vertices can then be translated back to obtain those of the original sets. An alternative definition of A and B, then, is the following:

$$A = \text{conv}\left\{x \in \mathbb{Z}^2 : x_1 - \rho x_2 \le \phi, x_2 \ge 0\right\},$$
$$B = \text{conv}\left\{x \in \mathbb{Z}^2 : x_1 - \rho x_2 \ge \phi, x_2 \ge 0\right\}.$$

It turns out that the vertices of A and B are related to the lattice-free split $\mathcal{S}$, given by

$$\mathcal{S} = \left\{x \in \mathbb{R}^2 : 0 \le x_1 - \lfloor \phi + \rho \rfloor x_2 \le 1\right\}. \tag{4}$$

Specifically, in the rest of this section, we discuss whether (i) the ray $\left(\begin{smallmatrix}\phi\\0\end{smallmatrix}\right) + \text{cone}\left(\begin{smallmatrix}\rho\\1\end{smallmatrix}\right)$ is parallel to the direction of the split $\mathcal{S}$ (Proposition 7, Fig. 2a), or (ii) the ray hits the boundary of $\mathcal{S}$ at an integer point (Proposition 8, Fig. 2b), or (iii) the ray hits the boundary of $\mathcal{S}$ at a fractional point on the B side (Proposition 9, Fig. 3), or (iv) the ray hits the boundary of $\mathcal{S}$ at a fractional point on the A side (Proposition 10).

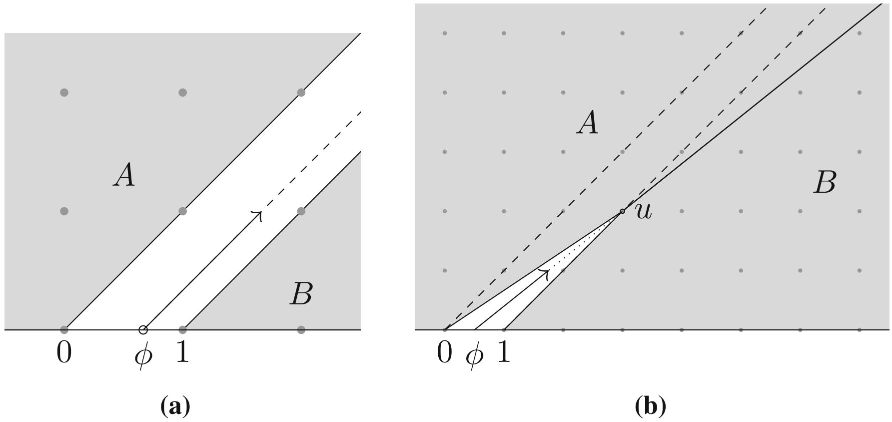

Fig. 2 Illustration of Propositions 7 and 8. **a** $\phi = \widehat{\phi + \rho}$. **b** $\phi \neq \widehat{\phi + \rho}$ and $u \in \mathbb{Z}^2$ ($\mathcal{S}$ shown in dashed lines)

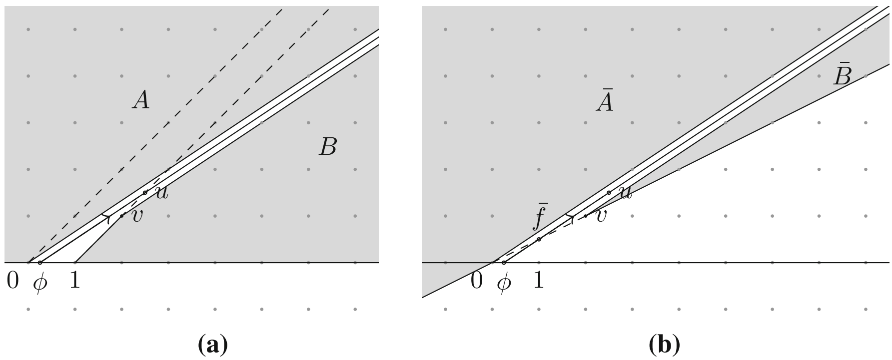

Fig. 3 Illustration of Proposition 9. **a** A, B, u, v and $\mathcal{S}$ (dashed lines). **b** $\bar{A}$ and $\bar{B}$

Note that $(0, 0)$ and $(1, 0)$ are always vertices of A and B, respectively. The next proposition shows that, in some cases, these are the only vertices of these two sets (Fig. 2a). This happens when the ray $\binom{\phi}{0} + \text{cone}\binom{\rho}{1}$ is parallel to the direction of the split $\mathcal{S}$.

Proposition 7 *If* $\phi = \widehat{\phi + \rho}$, *then* $\text{vert}(A) = \{\binom{0}{0}\}$ *and* $\text{vert}(B) = \{\binom{1}{0}\}$.

Proof First, note that the condition on ϕ implies that $\rho \in \mathbb{Z}$. We can, therefore, round down the right-hand side of one of the inequalities that define A, to obtain

$$A = \text{conv}\left\{x \in \mathbb{Z}^2 : x_1 - \rho x_2 \leq 0, x_2 \geq 0\right\}.$$

Clearly, (0,0) is the only vertex of the linear relaxation of this set. Since the vertex is integral, then the linear relaxation coincides with its integer hull. We conclude that $(0, 0)$ is the only vertex of A. To prove that $(1, 0)$ is the only vertex of B, we proceed similarly. □

Now, we focus on the case where $\phi \neq \widehat{\phi + \rho}$, i.e., the ray $\binom{\phi}{0} + \text{cone}\binom{\rho}{1}$ hits the boundary of the split $\mathcal{S}$, and the previous proposition does not apply.

Let $u \in \mathbb{R}^2$ be that point where the ray $\binom{\phi}{0} + \text{cone}\binom{\rho}{1}$ meets the split. The next proposition shows that, when u is an integral point, the vertices of A and B can be also easily determined (see Fig. 2b).

Proposition 8 *If* $u \in \mathbb{Z}^2$ *then* $\text{vert}(A) = \{\binom{0}{0}, u\}$ *and* $\text{vert}(B) = \{\binom{1}{0}, u\}$.

Proof First, we prove that $u_2x_1 - u_1x_2 \leq 0$ is a valid inequality for A. We assume that the ray hits the boundary of the split on the "B-side", i.e., on the line $x_1 - \lfloor \phi + \rho \rfloor x_2 = 1$. The other case is analogous. Let $x \in A \cap \mathbb{Z}^2$. Since x is not in the interior of the split, it must satisfy either $x_1 - \lfloor \phi + \rho \rfloor x_2 \leq 0$ or $x_1 - \lfloor \phi + \rho \rfloor x_2 \geq 1$. We prove that, in either case, $u_2x_1 - u_1x_2 \leq 0$.

First, suppose $x_1 - \lfloor \phi + \rho \rfloor x_2 \leq 0$. Since $u_2 \geq 0$, we can multiply both sides of this inequality by u_2 to obtain $u_2x_1 - \lfloor \phi + \rho \rfloor u_2x_2 \leq 0$. Also, since u is on the B-side boundary of the split, then $u_1 - \lfloor \phi + \rho \rfloor u_2 = 1$. Therefore, $-(u_1 - \lfloor \phi + \rho \rfloor u_2)x_2 \leq 0$. Summing the two previous inequalities, we obtain $u_2x_1 - u_1x_2 \leq 0$, as desired.

Now suppose $x_1 - \lfloor \phi + \rho \rfloor x_2 \geq 1$. Since u satisfies $u_1 - \rho u_2 = \phi$ and $u_1 - \lfloor \phi + \rho \rfloor u_2 = 1$, then we must have $u_1 = \frac{\rho - \phi \lfloor \phi + \rho \rfloor}{\rho - \lfloor \phi + \rho \rfloor}$, $u_2 = \frac{1 - \phi}{\rho - \lfloor \phi + \rho \rfloor}$. Let $\lambda_1 = \frac{\phi}{\rho - \lfloor \phi + \rho \rfloor}$ and $\lambda_2 = \frac{1}{\rho - \lfloor \phi + \rho \rfloor}$. Since u is on the B-side boundary of the split, we have $\phi < \widehat{\phi + \rho}$. That is, $\phi < \phi + \rho - \lfloor \phi + \rho \rfloor$, which implies $\rho - \lfloor \phi + \rho \rfloor > 0$. Since, by assumption, $0 < \phi < 1$, we conclude that $\lambda_1, \lambda_2 \geq 0$. Using the previous characterization of u, it is also straightforward to verify that, if we multiply the valid inequality $-x_1 + \lfloor \phi + \rho \rfloor x_2 \leq -1$ by λ_1, multiply the valid inequality $x_1 - \rho x_2 \leq \phi$ by λ_2, and then sum the resulting inequalities, we obtain $u_2x_1 - u_1x_2 \leq 0$, as desired.

Since $u_2x_1 - u_1x_2 \leq 0$ is valid, we may write

$$A = \text{conv}\left\{ x \in \mathbb{Z}^2 : \begin{array}{l} u_2x_1 - u_1x_2 \leq 0 \\ x_1 - \rho x_2 \leq \phi \\ x_2 \geq 0 \end{array} \right\}.$$

It is not hard to see that $(0, 0)$ and u are the only vertices of the linear relaxation of this set. Since the linear relaxation has integer vertices, it coincides with its integer hull. We conclude that $(0, 0)$ and u are the only vertices of A. The proof for $\text{vert}(B)$ is similar. □

Now we consider two more interesting cases, when $u \notin \mathbb{Z}^2$. In the first case, illustrated in Fig. 3, the ray hits the boundary of the split on the "B-side", i.e., on the line $x_1 - \lfloor \phi + \rho \rfloor x_2 = 1$. The next proposition describes two sets $\bar{A}$ and $\bar{B}$ such that $\text{vert}(A) = \text{vert}(\bar{A})$ and $\text{vert}(B) = \{\binom{1}{0}\} \cup \text{vert}(\bar{B})$. That is, in order to enumerate the vertices of A and B, it is sufficient to enumerate the vertices of $\bar{A}$ and $\bar{B}$, then add the point $\binom{1}{0}$. The vertices of $\bar{A}$ and $\bar{B}$ can be enumerated recursively, as we will see later.

Proposition 9 *Suppose* $u \notin \mathbb{Z}^2$ *and* $\phi < \widehat{\phi + \rho}$. *Let* v *be the lattice point closest to* u *in the segment between* u *and* $\binom{1}{0}$. *Let* $\bar{f}$ *be the intersection between the segment connecting* $\binom{0}{0}$ *to* v, *and the segment connecting* $\binom{\phi}{0}$ *and* u. *Define*

$$\bar{A} = \text{conv}\left(\mathbb{Z}^2 \cap \left(\bar{f} + \text{cone}\left\{\binom{\rho}{1}, \binom{0}{0} - \bar{f}\right\}\right)\right)$$
$$\bar{B} = \text{conv}\left(\mathbb{Z}^2 \cap \left(\bar{f} + \text{cone}\left\{\binom{\rho}{1}, v - \bar{f}\right\}\right)\right)$$

Then $\text{vert}(A) = \text{vert}(\bar{A})$ *and* $\text{vert}(B) = \left\{\binom{1}{0}\right\} \cup \text{vert}(\bar{B})$.

Proof (i) First, observe that $\bar{A}$ can be written as

$$\bar{A} = \text{conv}\left\{x \in \mathbb{Z}^2 : \begin{array}{l} x_1 - \rho x_2 \leq \phi \\ v_2 x_1 - v_1 x_2 \leq 0 \end{array}\right\}.$$

Also, because the triangle $\binom{0}{0}, \binom{\phi}{0}, \bar{f}$ is contained in the split $\mathcal{S}$, and thus contains no integral points, the inequality $v_2x_1 - v_1x_2 \leq 0$ can be added to the description of the set A, without affecting its definition:

$$A = \text{conv}\left\{x \in \mathbb{Z}^2 : \begin{array}{l} x_1 - \rho x_2 \leq \phi \\ v_2 x_1 - v_1 x_2 \leq 0 \\ x_2 \geq 0 \end{array}\right\}.$$

Thus, it becomes clear that $A \subseteq \bar{A}$.

In the following, we prove that $\text{vert}(A) = \text{vert}(\bar{A})$. Let $x \in \mathbb{R}^2$. First, we prove that, if $x \notin \text{vert}(A)$, then $x \notin \text{vert}(\bar{A})$. We may assume $x \in \bar{A}$, since otherwise x is clearly not a vertex of $\bar{A}$, and there is nothing to prove. We have two cases. In the first case, suppose $x \in A$. Since x is not a vertex of A, there exist $y^1, y^2 \in A \setminus \{x\}$ such that $x = \frac{1}{2}y^1 + \frac{1}{2}y^2$. Since $A \subseteq \bar{A}$, then $y^1, y^2 \in \bar{A}$, and we conclude that x is also not a vertex of $\bar{A}$. In the second case, suppose $x \notin A$. This implies $x_2 < 0$. Let $y^1 = x - \varepsilon v$, $y^2 = x + \varepsilon v$, where $\varepsilon = -\frac{x_2}{v_2} > 0$. Note that $y^1 \in \bar{A}$, since

$$y^1 = \underbrace{x}_{\in \bar{A}} + \underbrace{\varepsilon}_{\geq 0} \underbrace{(-v)}_{\in \text{rec}(\bar{A})}.$$

We also have $y^2 \in \bar{A}$, since

$$y^2 = \underbrace{\begin{pmatrix}0\\0\end{pmatrix}}_{\in \bar{A}} + \underbrace{\left(-x_1 + \frac{v_1 x_2}{v_2}\right)}_{\geq 0} \underbrace{\begin{pmatrix}-1\\0\end{pmatrix}}_{\in \text{rec}(\bar{A})}.$$

Since $y^1, y^2 \in \bar{A}$ and $x = \frac{1}{2}y^1 + \frac{1}{2}y^2$, we conclude, also in this case, that $x \notin \text{vert}(\bar{A})$.

Now we prove that, if $x \notin \text{vert}(\bar{A})$, then $x \notin \text{vert}(A)$. Similarly, we may assume $x \in A$, otherwise x is clearly not a vertex of A, and there is nothing to prove. Since $A \subseteq \bar{A}$, this also implies that $x \in \bar{A}$. We have two cases. In the first case, suppose $x_2 = 0$. Since x satisfies $v_2x_1 - v_1x_2 \leq 0$, then $x_1 \leq 0$. Since x is not a vertex of $\bar{A}$, then $x \neq \binom{0}{0}$, and $x_1 < 0$. Therefore,

$$x = \underbrace{\begin{pmatrix}0\\0\end{pmatrix}}_{\in A} + \underbrace{(-x_1)}_{>0} \underbrace{\begin{pmatrix}-1\\0\end{pmatrix}}_{\in \text{rec}(A)}.$$

We conclude that x is not a vertex of A. In the second case, suppose $x_2 > 0$. Since x is not a vertex of $\bar{A}$, there exists $d \in \mathbb{R}^2 \setminus \{0\}$ such that $x + d, x - d \in \bar{A}$. Let $y^1 = x + \varepsilon d$, $y^2 = x - \varepsilon d$, for some $\varepsilon > 0$. If ε is small enough, we have $y_2^1, y_2^2 \geq 0$, which implies $y^1, y^2 \in A$. Since $x = \frac{1}{2}y^1 + \frac{1}{2}y^2$, we conclude that x is not a vertex of A.

(ii) First, observe that B and $\bar{B}$ can be written as

$$B = \operatorname{conv}\left\{x \in \mathbb{Z}^2 : \begin{array}{l} x_1 - \rho x_2 \geq \phi \\ x_2 \geq 0 \end{array}\right\},$$
$$\bar{B} = \operatorname{conv}\left\{x \in \mathbb{Z}^2 : \begin{array}{l} x_1 - \rho x_2 \geq \phi \\ v_2 x_1 - v_1 x_2 \leq 0 \\ x_2 \geq 0 \end{array}\right\}.$$

Thus, it is clear that $\bar{B} \subseteq B$. In the following, we prove that $\operatorname{vert}(B) = \{\binom{1}{0}\} \cup \operatorname{vert}(\bar{B})$. Let $x \in \mathbb{R}^2$. First, we prove that if $x \notin \operatorname{vert}(B)$ then $x \neq \binom{1}{0}$ and $x \notin \operatorname{vert}(\bar{B})$. It is easy to see that $\binom{1}{0} \in \operatorname{vert}(B)$, thus $x \neq \binom{1}{0}$. Furthermore, if $x \notin \bar{B}$, then x is clearly not a vertex of $\bar{B}$ and we done, so we assume $x \in \bar{B}$. We have two cases. In the first case, suppose $v_2 x_1 - v_1 x_2 = 0$. We can prove that there exists $\varepsilon > 0$ such that

$$x = \underbrace{v}_{\in \bar{B}} + \underbrace{\varepsilon}_{>0} \underbrace{v}_{\in \operatorname{rec}(\bar{B})}.$$

Therefore, x is not a vertex of $\bar{B}$. In the second case, suppose $v_2 x_1 - v_1 x_2 < 0$. Since x is not a vertex of B, there exists $d \in \mathbb{R}^2 \setminus \{0\}$ such that $x + d, x - d \in B$. For a small enough $\varepsilon > 0$, we can prove that $x + \varepsilon d, x - \varepsilon d \in \bar{B}$. In either case, we conclude that x is not a vertex of $\bar{B}$.

Now we prove that, if $x \neq \binom{1}{0}$ and $x \notin \operatorname{vert}(\bar{B})$, then $x \notin \operatorname{vert}(B)$. Clearly, if $x \notin B$ then $x \notin \operatorname{vert}(B)$ and we are done, so we assume $x \in B$. Once again, we have two cases. In the first case, suppose $x \in \bar{B}$. Since $x \notin \operatorname{vert}(\bar{B})$, there exist $y^1, y^2 \in \bar{B} \setminus \{x\}$ such that $x = \frac{1}{2}y^1 + \frac{1}{2}y^2$. Since $\bar{B} \subseteq B$, then $y^1, y^2 \in B$. Therefore, x is not a vertex of B. In the second case, suppose $x \notin \bar{B}$. Since $x \neq \binom{1}{0}$, it is clear that, if $x_2 = 0$, then x is not a vertex of B and we are done. Therefore, we assume $x_2 > 0$. Let

$$y^1 = x + \epsilon\left[v - \binom{1}{0}\right], \; y^2 = x - \epsilon\left[v - \binom{1}{0}\right],$$

where $\epsilon > 0$. It is not hard to prove that, for a small enough ϵ, we have $y^1, y^2 \in B$. Since $x = \frac{1}{2}y^1 + \frac{1}{2}y^2$, we conclude that, in any case, $x \notin \operatorname{vert}(B)$.

In the case where the ray hits the boundary of the split on the "A-side", i.e., on the line $0 = x_1 - \lfloor \phi + \rho \rfloor x_2$, we have a similar result. The next proposition describes two sets $\bar{A}$ and $\bar{B}$ such that $\operatorname{vert}(A) = \{\binom{0}{0}\} \cup \operatorname{vert}(\bar{A})$ and $\operatorname{vert}(B) = \operatorname{vert}(\bar{B})$. We skip its proof since it is analogous to the proof of Proposition 9.

Proposition 10 *Suppose $u \notin \mathbb{Z}^2$ and $\phi > \widehat{\phi + \rho}$. Let v be the lattice point closest to u in the segment between u and $\binom{0}{0}$. Let $\bar{f}$ be the intersection between the segment connecting $\binom{1}{0}$ to v, and the segment connecting $\binom{\phi}{0}$ and u. Define*

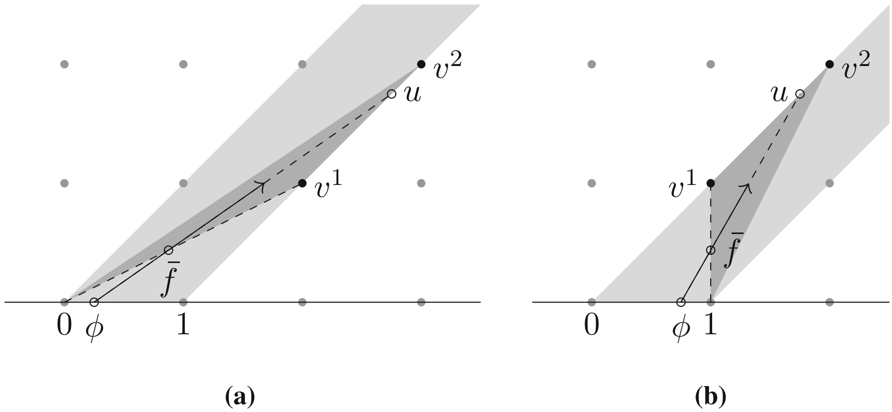

Fig. 4 How $\bar{f}$, u, v^1 and v^2 are found in Proposition 11. **a** $\phi < \widehat{\phi + \rho}$. **b** $\phi > \widehat{\phi + \rho}$

$$\bar{A} = \operatorname{conv}\left(\mathbb{Z}^2 \cap \left(\bar{f} + \operatorname{cone}\left\{\binom{\rho}{1}, v - \bar{f}\right\}\right)\right)$$
$$\bar{B} = \operatorname{conv}\left(\mathbb{Z}^2 \cap \left(\bar{f} + \operatorname{cone}\left\{\binom{\rho}{1}, \binom{0}{0} - \bar{f}\right\}\right)\right)$$

Then $\operatorname{vert}(A) = \left\{\binom{0}{0}\right\} \cup \operatorname{vert}(\bar{A})$ *and* $\operatorname{vert}(B) = \operatorname{vert}(\bar{B})$.

Now the only question remaining is how to compute the vertices of $\bar{A}$ and $\bar{B}$. The next proposition shows that this can be done recursively. By applying an appropriate affine integral unimodular transformation to the coordinate system and scaling of the rays, the sets $\bar{A}$ and $\bar{B}$ can be written in the same form as the original sets A and B. Therefore, the vertices of $\bar{A}$ and $\bar{B}$ can be obtained by recursively applying Propositions 7–10.

Proposition 11 *Suppose $u \notin \mathbb{Z}^2$. If $\phi < \widehat{\phi + \rho}$, let $\bar{A}$ and $\bar{B}$ be defined as in Proposition 9. If $\phi > \widehat{\phi + \rho}$, let $\bar{A}$ and $\bar{B}$ be defined as in Proposition 10. In either case, there exist $\bar{\phi}, \bar{\rho} \in \mathbb{R}$ and an affine integral unimodular transformation $\tau : \mathbb{R}^2 \to \mathbb{R}^2$ such that*

$$\tau(\bar{A}) = \operatorname{conv}\left(\mathbb{Z}^2 \cap \left(\begin{pmatrix}\bar{\phi}\\0\end{pmatrix} + \operatorname{cone}\left\{\begin{pmatrix}\bar{\rho}\\1\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}\right\}\right)\right),$$
$$\tau(\bar{B}) = \operatorname{conv}\left(\mathbb{Z}^2 \cap \left(\begin{pmatrix}\bar{\phi}\\0\end{pmatrix} + \operatorname{cone}\left\{\begin{pmatrix}\bar{\rho}\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right\}\right)\right).$$

Proof Suppose $\phi < \widehat{\phi + \rho}$. Let $\bar{f}$ and $v \in \mathbb{Z}^2$ as defined in Proposition 9. Let $v^1 = v$ and let v^2 be the lattice point closest to u in the half-line $u + \lambda(u - \binom{1}{0})$, $\lambda \geq 0$. That is, v^1 and v^2 are the closest lattice points to u in the line passing through u and $\binom{1}{0}$ (see Fig. 4a). Let $\tau : \mathbb{R}^2 \to \mathbb{R}^2$ be an affine function such that $\tau\binom{0}{0} = \binom{0}{0}$, $\tau(v^1) = \binom{1}{0}$, $\tau(v^2) = \binom{1}{1}$. Such a transformation exists, since v^1 and v^2 are linearly independent. Furthermore, it is integral and unimodular, since the triangle defined by $\binom{0}{0}$, v^1 and v^2 has integral vertices and its area equals $\frac{1}{2}$. Therefore,

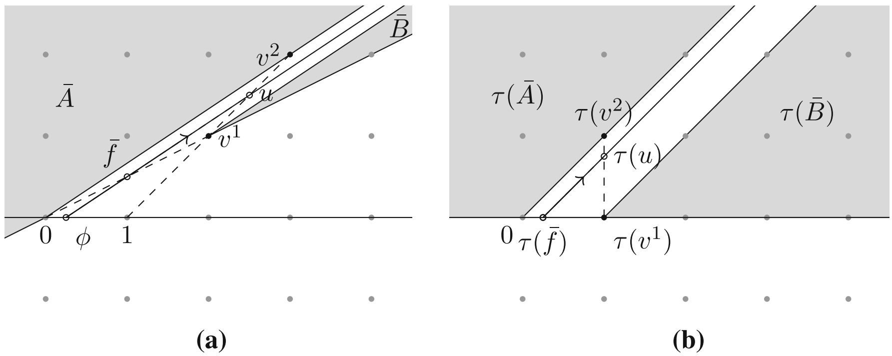

Fig. 5 Transformation τ of Proposition 11. **a** Figure 3b zoomed in. **b** Transformed by τ

$$\tau(\bar{A}) = \text{conv}\left(\mathbb{Z}^2 \cap \left(\tau(\bar{f}) + \text{cone}\left\{\tau\begin{pmatrix}\rho\\1\end{pmatrix}, \tau\left(\begin{pmatrix}0\\0\end{pmatrix} - \bar{f}\right)\right\}\right)\right)$$
$$\tau(\bar{B}) = \text{conv}\left(\mathbb{Z}^2 \cap \left(\tau(\bar{f}) + \text{cone}\left\{\tau\begin{pmatrix}\rho\\1\end{pmatrix}, \tau(v^1 - \bar{f})\right\}\right)\right)$$

Since $\bar{f} \in \text{conv}\left\{\binom{0}{0}, v^1\right\}$, then $\tau(\bar{f}) \in \text{conv}\left\{\binom{0}{0}, \binom{1}{0}\right\}$, which implies that there exists $\bar{\phi} \in \mathbb{R}$ such that $\tau(\bar{f}) = \begin{pmatrix}\bar{\phi}\\0\end{pmatrix}$. Furthermore, it is not hard to see that there exist $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}_+$ such that

$$\lambda_1 \tau\begin{pmatrix}\rho\\1\end{pmatrix} = \begin{pmatrix}\bar{\rho}\\1\end{pmatrix}, \lambda_2 \tau\left(\begin{pmatrix}0\\0\end{pmatrix} - \bar{f}\right) = \begin{pmatrix}-1\\0\end{pmatrix}, \lambda_3 \tau\left(v^1 - \bar{f}\right) = \begin{pmatrix}1\\0\end{pmatrix}$$

This concludes the proof for this case (see Fig. 5). When $\phi > \widehat{\phi + \rho}$, the proof is similar, constructing v^1 and v^2 in an analogous way (see Fig. 4b), but we let τ be an affine function satisfying $\tau\binom{1}{0} = \binom{1}{0}, \tau(v^1) = \binom{0}{0}, \tau(v^2) = \binom{0}{1}$, instead. □

Using Propositions 7–10, we now have a complete recursive algorithm for computing the vertices of A and B. The first step is to verify whether $\phi = \widehat{\phi + \rho}$. If so, the vertices are given by Proposition 7. If not, we construct the split $\mathcal{S}$ and verify whether the intersection of its boundary with the ray $\binom{\phi}{0} + \text{cone}\binom{\rho}{1}$ is an integral point. If so, the vertices of A and B are given by Proposition 8. If not, then either Propositions 9 or 10 apply, in which case the vertices of A and B are the same as the vertices of $\bar{A}$ and $\bar{B}$, in addition to either $\binom{0}{0}$ or $\binom{1}{0}$. In order to compute the vertices of $\bar{A}$ and $\bar{B}$, we proceed recursively. First, we apply the transformation described in Proposition 11, so that $\bar{A}$ and $\bar{B}$ are written in the same form as the original sets A and B, then we repeatedly apply Propositions 7–10. A non-recursive version of this algorithm is described in Algorithm 12.

Now that we have an algorithm for enumerating the vertices of A and B, we finish this section by describing how can we use the previous propositions to get a complete list of maximal S-free sets that induce facets of P_I. Definition 13 describes the

Algorithm 12 Non-recursive algorithm for enumerating the vertices of A, B

function ENUMERATEVERTICES(ϕ, ρ)
    $U \leftarrow I, t \leftarrow \mathbf{0}$
    $X^A \leftarrow \{(0,0)\}, X^B \leftarrow \{(1,0)\}$
    loop
        if $\phi == \widehat{\phi + \rho}$ **then**
            return X^A, X^B
        else if $\phi < \widehat{\phi + \rho}$ **then**
            $u \leftarrow \left(\frac{\rho - \phi\lfloor\phi+\rho\rfloor}{\rho - \lfloor\phi+\rho\rfloor}, \frac{1-\phi}{\rho-\lfloor\phi+\rho\rfloor}\right)$
            if $u \in \mathbb{Z}^2$ **then**
                return $X^A \cup \{Uu + t\}, X^B \cup \{Uu + t\}$
            $v^1 \leftarrow (1 + \lfloor\phi+\rho\rfloor \lfloor u_2 \rfloor, \lfloor u_2 \rfloor)$
            $v^2 \leftarrow (1 + \lfloor\phi+\rho\rfloor \lceil u_1 \rceil, \lceil u_1 \rceil)$
            $X^B \leftarrow X^B \cup \{Uv^1 + t\}$
            $W \leftarrow \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v^1_1 & v^2_1 \\ v^1_2 & v^2_2 \end{bmatrix}^{-1}$
            $y \leftarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
            $f' \leftarrow \left(\frac{v^1_2 \phi}{v^1_1 - v^1_2 \rho}, \frac{v^1_1 \phi}{v^1_1 - v^1_2 \rho}\right)$
        else if $\phi > \widehat{\phi + \rho}$ **then**
            $u \leftarrow \left(\frac{\phi\lfloor\phi+\rho\rfloor}{\lfloor\phi+\rho\rfloor - \rho}, \frac{\phi}{\lfloor\phi+\rho\rfloor-\rho}\right)$
            if $u \in \mathbb{Z}^2$ **then**
                return $X^A \cup \{Uu + t\}, X^B \cup \{Uu + t\}$
            $v^1 \leftarrow (\lfloor\phi+\rho\rfloor \lfloor u_2 \rfloor, \lfloor u_2 \rfloor)$
            $v^2 \leftarrow (\lfloor\phi+\rho\rfloor \lceil u_1 \rceil, \lceil u_1 \rceil)$
            $X^A \leftarrow X^A \cup \{Uv^1 + t\}$
            $W \leftarrow \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v^1_1 - 1 & v^2_1 - 1 \\ v^1_2 & v^2_2 \end{bmatrix}^{-1}$
            $y \leftarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} - W \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
            $f' \leftarrow \left(\frac{\phi(v^1_1 - 1) - \rho v^1_2}{\rho v^1_2 - (v^1_1 - 1)}, \frac{(1-\phi) v^1_2}{\rho v^1_2 - (v^1_1 - 1)}\right)$
        $f \leftarrow W f' + y, \phi \leftarrow f_1$
        $r \leftarrow W \binom{\rho}{1} + y, \rho \leftarrow \frac{r_1}{r_2}$
        $t \leftarrow t - UW^{-1} y$
        $U \leftarrow UW^{-1}$

sequence of S-free sets that we construct, one per iteration of the vertex enumeration algorithm. Note that the definition is recursive. Given ϕ and ρ, Propositions 7–10 show how to compute one S-free set (either $\mathcal{S}$ itself, or W^u if u exists). Then, if u exists and $u \notin \mathbb{Z}^2$, Proposition 11 provides an affine transformation τ and a new model, determined by $\bar{\phi}$ and $\bar{\rho}$, which will yield further S-free sets. The sequence $\mathcal{W}(\phi, \rho)$ is constructed by concatenating W^u and the subsequent S-free sets $\mathcal{W}(\bar{\phi}, \bar{\rho})$ given by the new model, suitably transformed back into the original space. Because the sequence is ordered, we use the angle bracket notation to construct the sequence from its elements, i.e., $\mathcal{W}(\phi, \rho) = \langle W_1, \ldots, W_k \rangle$, where k is the length of the sequence.

Definition 13 Let $\mathcal{W}(\phi, \rho)$ be a sequence of sets defined as follows:

(i) If the conditions of Proposition 7 are satisfied, then $\mathcal{W}(\phi, \rho) = \langle \mathcal{S} \rangle$, where $\mathcal{S}$ is the split defined in (4).

(ii) If the conditions of Proposition 8 are satisfied, then $W(\phi, \rho) = \langle W^u \rangle$, where

$$W^u = u + \text{cone}\left\{\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right) - u, \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) - u\right\}.$$

(iii) Suppose that the conditions of either Propositions 9 or 10 are satisfied. Let $\bar{\phi}, \bar{\rho}, \tau$ be defined as in Proposition 11, and let

$$\mathcal{W}(\bar{\phi}, \bar{\rho}) = \langle \bar{W}_1, \ldots, \bar{W}_\ell \rangle,$$

where ℓ is the length of the sequence $\mathcal{W}(\bar{\phi}, \bar{\rho})$. Then,

$$\mathcal{W}(\phi, \rho) = \left\langle W^u, \tau^{-1}(\bar{W}_1), \ldots, \tau^{-1}(\bar{W}_\ell) \right\rangle,$$

where W^u is defined as in (ii).

For every $j \in \{1, \ldots, k\}$, it is easy to see that W_j is tight at three integral points; either two vertices of A and one vertex of B, or two vertices of B and one vertex of A. Observe moreover that, for any combination of three vertices not generated in this fashion, one could not construct a corresponding S-free wedge: First, note that the two vertices belonging to the same side must be consecutive, otherwise the wedge cannot be S-free. Then, given a pair of tight vertices on one side, the S-free wedge that is tight at those vertices and a third on the other side is unique. For every pair of consecutive vertices of either A or B there is a wedge W_j that is tight for these vertices. W_j also has a vertex that is tight on the other side. If we replace this third vertex by any other, the other vertex will either be on the boundary or outside of the initial wedge. In the first case, the new wedge would be identical to the initial one, and in the second case, it would not be S-free.

The next proposition shows that W_j is also S-free. Then Theorem 6 implies that the intersection cut from W_j yields a facet-defining inequality for $\text{conv}(P_I)$. By Proposition 4, we now have a complete $\mathcal{H}$-representation of $\text{conv}(P_I)$.

Proposition 14 *Every set in $\mathcal{W}(\phi, \rho)$ is maximal and S-free.*

Proof We prove the claim by structural induction. If $\mathcal{W}(\phi, \rho) = \langle S \rangle$ or $\mathcal{W}(\phi, \rho) = \langle W^u \rangle$, then the proposition is clearly true. Now suppose $\mathcal{W}(\phi, \rho) = \left\langle W^u, \tau^{-1}(\bar{W}_1), \ldots, \tau^{-1}(\bar{W}_\ell) \right\rangle$, and suppose, by induction, that $\bar{W}_1, \ldots, \bar{W}_\ell$ are maximal S-free sets containing $\left(\begin{smallmatrix}\bar{\phi}\\0\end{smallmatrix}\right)$ in their interior. Clearly, W^u is maximal S-free and contains $\left(\begin{smallmatrix}\phi\\0\end{smallmatrix}\right)$. Let $j \in \{1, \ldots, k\}$. We prove that the same holds for $\bar{W}_j$. Since $\bar{W}_j$ is S-free, then $\tau^{-1}(W_j)$ does not contain any integral points above the line that connects $(0, 0)$ and v^1 (in the first case of Proposition 11) or $(1, 0)$ and v^1 (in the second case). Furthermore, the region of $\tau^{-1}(\bar{W}_j)$ that lies below the line is entirely contained in

W^u. Therefore, $\bar{W}_j$ is S-free. Since $\bar{W}_j$ is maximal, it is not hard to see that $\tau^{-1}(W_j)$ is also maximal. □

Finally, we determine the time complexity of Algorithm 12. Since we are in the unusual context of a fixed-dimension problem, it is worth specifying that we adopt the classic computational model of counting elementary arithmetic operations $(+, -, \times, /)$. This is the same computational model used by Harvey [36]. In that context, it is immediate to observe that Algorithm 12 is optimal (up to a constant factor), since its running time is linear in the size of the output: it generates one facet per iteration, and each iteration comprises a constant number of operations. In order to determine the size of the output in terms of the size of the input, we can use the bound of Cook et al. [18] on the number of vertices of an integer hull. This upper bound is $O(m^n \varphi^{n-1})$ where n is the dimension, m is the number of linear inequalities, and φ is the maximum encoding size of one inequality. In our case, $n = 2$, $m = 1$ and φ is the total encoding size for ρ and ϕ, so we perform $O(\varphi)$ iterations, and thus, $O(\varphi)$ operations. This is equivalent to the more general algorithm of Harvey [36]. However, our method is much simpler in terms of implementation, as it consists only in a few operations on 2-by-2 matrices.

Coming back to our objective of generating "more" one-row cuts, we can give some geometric intuition at this point. Both Gomory's mixed-integer (GMI) cut and the mixed-integer rounding (MIR) cut can be visualized as follows. Consider the initial untransformed model P_I and the intersection cut arising from the split set $\mathcal{S}$ (recall that $\mathcal{S}$ is tight at $(\lfloor \phi \rfloor, 0)$, $(\lceil \phi \rceil, 0)$, $(\lfloor \phi + \rho \rfloor, 1)$ and $(\lceil \phi + \rho \rceil, 1)$). This intersection cut is identical to the one that is obtained from the first wedge generated by Algorithm 12, since they share the same intersection points. It can be observed, by comparing the coefficients, that this inequality corresponds to both the GMI formula and the MIR cut for P_I. It could equivalently be obtained by computing the GMI cut for a relaxation of P_I where s_1 is continuous, then applying a lifting argument to take into account the integrality of s_1, thus strengthening the cut coefficient for s_1. On the other hand, the intersection cuts generated at all subsequent iterations of Algorithm 12 are distinct from the GMI cut, and cannot be obtained through lifting, since they are also facet-defining for $\text{conv}(P_I)$, yet have weaker coefficients for s_2 and s_3.

Furthermore, again by coefficient inspection, one can show that the 2-step MIR inequality [21] can be derived as an inequality for $\text{conv}(P_I)$, and in fact it is the inequality obtained once Algorithm 12 switches from using Proposition 9 to using Proposition 10 (or vice-versa) for the first time. We again skip a formal derivation here as the process is lengthy yet straightforward.

The fact that 2-step MIR cuts have a split rank of 2 prompted us to study the split rank of $\text{conv}(P_I)$, which we do in the next section.

4 Upper bound on the split rank

In this section, we prove that the split rank of $\text{conv}(P_I)$ is at most the sum of the number of vertices of A and the number of vertices of B. Note that the split rank of an inequality is the smallest nonnegative integer k such that the inequality is implied by the kth split *closure*, as defined by Cook et al. [17].

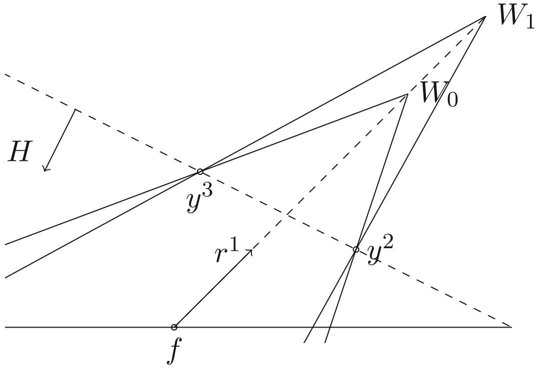

Fig. 6 Wedges W_0 and W_1 in the configuration of Lemma 15

In the following, let

$$P_{LP} = \left\{(x, s) \in \mathbb{R}^2 \times \mathbb{R}^3_+ : \binom{x_1}{x_2} = \binom{\phi}{0} + \binom{\rho}{1} s_1 + \binom{1}{0} s_2 + \binom{-1}{0} s_3\right\}$$

be the linear relaxation of P_I. Recall that in our notation, $f = \binom{\phi}{0}$ and $r^1 = \binom{\rho}{1}$. In order to prove our result, we first need Lemma 15. It shows that given two wedges in a specific configuration and their induced intersection cuts, there is a half-plane where any point cut off by one is cut off by the other.

Lemma 15 *Let W_0 and W_1 be two distinct wedges with apex on $f + \text{cone}(r^1)$ and containing f in their relative interiors. Let $(\alpha^0)^T s \geq 1$ and $(\alpha^1)^T s \geq 1$ be the intersection cuts obtained from W_0 and W_1, respectively. Suppose that $\alpha^1_1 < \alpha^0_1$, and that the boundaries of W_0, W_1 intersect at two distinct points $y^2, y^3 \in \mathbb{R}^2$. Let $H \subseteq \mathbb{R}^2$ be the closed half-space that contains y^2, y^3 in its boundary and does not contain the apices of W_0, W_1* (Fig. 6)*. If $f \in \text{int}(H)$, then, for any $(\bar{x}, \bar{s}) \in P_{LP}$ such that $\bar{x} \in \text{int}(H)$ and ${\alpha^1}^T \bar{s} < 1$, we also have ${\alpha^0}^T \bar{s} < 1$.*

Proof First, we construct $\alpha^h \in \mathbb{R}^3$ such that, for any $(\bar{x}, \bar{s}) \in P_{LP}$, we have $\bar{x} \in \text{int}(H)$ if and only if $(\alpha^h)^T \bar{s} < 1$. The statement $\bar{x} \in \text{int}(H)$ is equivalent to $c^T \bar{x} < d$ for some $c \in \mathbb{R}^2$, $d \in \mathbb{R}$. Recalling that $\bar{x} = f + R\bar{s}$, this is equivalent to $c^T(f + R\bar{s}) < d$, i.e., $(c^T R)\bar{s} < d - c^T f$. Since f belongs to the interior of H, we know that $c^T f < d$, so the right-hand side of the previous equation is positive. Dividing by that right-hand side, we obtain the desired form $(\alpha^h)^T \bar{s} < 1$ where $\alpha^h = \frac{c^T R}{d - c^T f}$.

Next, we prove that α^0 is a convex combination of α^1 and α^h. Consider the three lines $\alpha^0_1 s_1 + \alpha^0_2 s_2 = 1$, $\alpha^1_1 s_1 + \alpha^1_2 s_2 = 1$ and $\alpha^h_1 s_1 + \alpha^h_2 s_2 = 1$. Note that each of these correspond to one face of each of W_0 and W_1 and H, so they intersect in a single point y^2. Therefore, $(\alpha^0_1, \alpha^0_2) = \lambda^2(\alpha^h_1, \alpha^h_2) + (1 - \lambda^2)(\alpha^1_1, \alpha^1_2)$, for some $\lambda^2 \in \mathbb{R}$. Similarly, for the other intersection y^3, we obtain $(\alpha^0_1, \alpha^0_3) = \lambda^3(\alpha^h_1, \alpha^h_3) + (1 - \lambda^3)(\alpha^1_1, \alpha^1_3)$. for some $\lambda^3 \in \mathbb{R}$. Together, these relationships show $\lambda^2 = \lambda^3$. Let $\lambda := \lambda^2 = \lambda^3$, we get $\alpha^0 = \lambda\alpha^h + (1 - \lambda)\alpha^1$. Since $\alpha^1_1 < \alpha^0_1$ and H does not contain the apex of W_0 or W_1, we have that $\alpha^1_1 < \alpha^0_1 < \alpha^h_1$, so α^0 is not only a linear combination of α^h and α^1, but also a convex combination (i.e., $0 \le \lambda \le 1$). Therefore, ${\alpha^h}^T s < 1$ and ${\alpha^1}^T s < 1$ together imply ${\alpha^0}^T s < 1$. □

Let $W_1, \dots, W_k$ be as defined in Sect. 3. The next theorem shows that the intersection cut from W_j has a split rank at most j.

Theorem 16 *For every $j \in \{1, \dots, k\}$, the intersection cut from W_j has split rank at most j with respect to P_{LP}.*

Proof We prove the claim by induction. The first wedge W_1 has the same intersection points as the split $\mathcal{S}$, so the corresponding cut has split rank 1. Let $j \in \{2, \dots, k\}$ and assume now that W_{j-1} yields a cut of split rank $j-1$ or less. In the following, we prove that the intersection cut generated from W_j is implied by the intersection cut generated from W_{j-1} together with a split cut, and, therefore, W_j yields a cut with split rank at most j. We assume that W_j is not a split, otherwise there is nothing to prove. Note that W_{j-1} and W_j are in the same configuration as wedges W_0 and W_1 of Lemma 15. Let $(x, s) \in P_{LP}$ be a point that does not satisfy the intersection cut from wedge W_j. This implies that $x \in \text{int}(W_j)$. If $x \in \text{int}(H)$, we apply Lemma 15 to show that (x, s) also does not satisfy the cut from wedge W_{j-1}. If $x \notin \text{int}(H)$, then x belongs to the interior of the split that was considered when generating W_j, hence (x, s) does not satisfy the intersection cut obtained from this split. We conclude that the intersection cut obtained from W_j is implied by the cut from W_{j-1} together with a split cut. Since the cut obtained from W_j has split rank at most $j-1$, then the cut obtained from W_j has split rank at most j.

Corollary 17 *Let k_2 and k_3 be the number of vertices of* $\text{conv}(K_2)$ *and* $\text{conv}(K_3)$*, respectively. The split rank of* $\text{conv}(P_I)$ *is at most* $k_2 + k_3 - 1$.

5 Multiple integral variables via lifting

We now consider how to obtain valid inequalities for the single-row corner relaxation when the integrality of multiple non-basic variables is preserved. One approach, using the same idea from Sect. 2, is to study the facial structure of a continuous $(m+1)$-row model, where m is the number of integral non-basic variables. Unfortunately, when considering three or more rows, this relaxation is significantly more complex, and not as well understood. Therefore, we focus instead on *lifting* the valid inequalities we obtained in Sect. 2. Given an S-free set

$$B = \left\{ x \in \mathbb{R}^2 : g_i^T (x - f) \le 1, \quad i = 1, \dots, k \right\},$$

we recall that the intersection cut obtained from B is given by

$$\psi \begin{pmatrix} \rho \\ 1 \end{pmatrix} s_1 + \psi \begin{pmatrix} 1 \\ 0 \end{pmatrix} s_2 + \psi \begin{pmatrix} -1 \\ 0 \end{pmatrix} s_3 \ge 1, \tag{5}$$

where $\psi(r) = \max_{i=1,\dots,k} g_i^T r$. Inequality (5) is valid for $\text{conv}(P_I)$. In the following, let

$$P_I^+ := \left\{ (x, s, z) \in S \times \mathbb{R}_+^3 \times \mathbb{Z}_+^m : \binom{x_1}{x_2} = \binom{\phi}{0} + \binom{\rho}{1} s_1 + \binom{1}{0} s_2 + \binom{-1}{0} s_3 + \sum_{i=1}^{m} \binom{\mu_i}{0} z_i \right\},$$

where $m \in \mathbb{Z}_+$ and $\mu \in \mathbb{Q}^m$. Given a valid inequality for $\text{conv}(P_I)$ written as (5), we want to obtain a strong valid inequality for $\text{conv}(P_I^+)$. More precisely, we want to find a function $\pi : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\psi\binom{\rho}{1} s_1 + \psi\binom{1}{0} s_2 + \psi\binom{-1}{0} s_3 + \sum_{i=1}^{m} \pi\binom{u_i}{0} z_i \geq 1$$

is satisfied by every point in $\text{conv}(P_I^+)$. Such π is called a *lifting* of ψ.

For the case where $S = \mathbb{Z}^2$ and B is a lattice-free set, it is well known that if π is defined as

$$\pi(r) = \min_{k \in \mathbb{Z}^2} \psi(r + k) \tag{6}$$

for all $r \in \mathbb{R}^2$, then π is a lifting of ψ. This function, introduced by Gomory and Johnson [33], is called the *trivial lifting* of ψ. When B is a maximal lattice-free set, Fukasawa et al. [29] presented a constant-time algorithm for evaluating (6). When B is a type-2 or type-3 triangle, a closed formula for evaluating this function is also available [12].

In our case, where $S = \mathbb{Z} \times \mathbb{Z}_+$ and B is S-free, it is straightforward to establish that

$$\pi\binom{\mu}{0} := \min_{k_2 \in \mathbb{Z}_+} \min_{k_1 \in \mathbb{Z}} \psi\binom{\mu + k_1}{k_2} \tag{7}$$

is a lifting of ψ ("Appendix A"). In Algorithm 18, we present a finite algorithm that evaluates (7) for any $\mu \in \mathbb{R}$. The algorithm is a variation of the algorithm presented in [29] and, although not guaranteed to finish in constant time, it performed well during our computational experiments. Although its proof of correctness and finiteness are very similar to the ones presented in [29], we include them here for completeness. At each iteration of the main loop, we solve the problem

$$h(\bar{k}_2) = \min_{k_1 \in \mathbb{Z}} \psi\binom{\mu + k_1}{\bar{k}_2}$$

for some fixed value $\bar{k}_2$, starting from zero, and going up. We also keep track of the smallest optimal value found so far, in the variable η^*. The algorithm stops when $\bar{k}_2 > \frac{\eta^*}{\zeta}$ since, for every $\bar{k}_2$ satisfying this condition, we have

$$\min_{k_1 \in \mathbb{Z}} \psi\binom{\mu + k_1}{\bar{k}_2} \geq \min_{k_1 \in \mathbb{R}} \psi\binom{\mu + k_1}{\bar{k}_2} = \min_{\alpha \in \mathbb{R}} \psi\binom{\alpha}{\bar{k}_2} = \bar{k}_2 \min_{\alpha \in \mathbb{R}} \psi\binom{\alpha}{1} = \zeta \bar{k}_2 \geq \zeta \frac{\eta^*}{\zeta} = \eta^*.$$

Therefore, by considering such $\bar{k}_2$, the incumbent value η^* can never be improved. It can also be easily verified that, for large enough $\bar{k}_2$, the stopping condition is satisfied and, therefore, the algorithm always terminates.

Algorithm 18 Trivial Lifting

function TRIVIALLIFTING(μ)
  Let $g(\bar{\alpha}_2) := \min_{\alpha_1 \in \mathbb{R}} \psi \begin{pmatrix} \alpha_1 \\ \bar{\alpha}_2 \end{pmatrix}$
  Let $h(\bar{k}_2) := \min_{k_1 \in \mathbb{Z}} \psi \begin{pmatrix} \mu + k_1 \\ \bar{k}_2 \end{pmatrix}$
  $\zeta \leftarrow g(1)$
  $\eta^* \leftarrow h(0)$
  $\bar{k}_2 \leftarrow 1$
  repeat
    $\eta^* \leftarrow \min \{\eta^*, h(\bar{k}_2)\}$
    $\bar{k}_2 \leftarrow \bar{k}_2 + 1$
  until $\bar{k}_2 > \frac{\eta^*}{\zeta}$
  return η^*

6 Computational experiments

In order to evaluate the strength of wedge cuts, we implemented a cut generator and tested it on the benchmark set of the MIPLIB 2010. We measured the gap closed by the inclusion of wedge cuts and compared it to the gap closed by considering MIR cuts alone. We also evaluated the speed of the trivial lifting algorithm presented in Sect.5.

The cut generator performed the following steps. First, the linear relaxation of the presolved problem was solved, and a certain basic solution with value z_{LP} was obtained. The optimal tableau was stored. Although we solved the relaxation again at a later time, we always used this first optimal tableau to generate all the cuts, hence obtaining only rank-1 cuts. Next, for each row of the tableau corresponding to an integral basic variable, an MIR cut was generated and added to the problem. The strengthened relaxation was then solved again, and another basic solution x^{MIR}, with value z_{MIR} was obtained. Then, every possible wedge cut was generated and added to the problem, provided that it cut off the previous solution x^{MIR}. More precisely, for each row of the tableau corresponding to an integral basic variable, and for each integral non-basic variable x_i that has non-zero coefficient in that row, we identified the coefficient corresponding to x_i with ρ, and generated all the facet-defining wedge and split cuts, as described in Sect.2. The cut coefficients for the remaining integral non-basic variables was calculated according to the algorithm from Sect. 5. Finally, the relaxation was solved again, and a basic solution with value z_W was obtained. In the following, we also denote by z_{OPT} the value of the optimal solution for the original mixed-integer problem.

The cut generator was implemented in C++ and compiled with the GNU C++ Compiler 4.8.4. The source code has been made available online [30]. For the LP solver, we used the library IBM ILOG CPLEX 12.6.2. Considerable care was taken to avoid the generation of invalid cuts. CPLEX was configured for numerical emphasis, and once the LP was solved, each double-precision floating point entry of the resulting tableau was converted to an exact rational number. To avoid the propagation of floating point errors, the enumeration of the facets of the knapsack sets was performed using exact arithmetic, with the help of the GNU Multiple Precision Arithmetic Library 6.1.0 [35]. The cut coefficients were then converted back to double-precision floating point numbers and given to CPLEX. We discarded all cuts with high coefficient dynamism (ratio between the magnitudes of largest and the smallest coefficients of 10^6 or larger), then considered only the remaining inequalities that cut off the fractional solution x^{MIR} by a significant amount (10^{-6} or more).

Our testbed was the benchmark set of the MIPLIB 2010, which is composed by 87 instances of real-world mixed integer programs. For each instance, the following performance indicators were computed:

- ORIG-GAP, the original gap between the first linear relaxation and the original mixed-integer program:

$$\frac{z_{OPT} - z_{LP}}{|z_{OPT}|}$$

- MIR-PERF, the amount of the original gap that was closed by the inclusion of the MIR inequalities:

$$\frac{z_{MIR} - z_{LP}}{z_{OPT} - z_{LP}}$$

- W-PERF, the amount of the original gap that what was closed by the inclusion of all the wedge inequalities:

$$\frac{z_W - z_{LP}}{z_{OPT} - z_{LP}}$$

- W-REL, the contribution of the wedge cuts to the gap closure; that is, the amount of the original gap that was closed by wedge inequalities which are not equivalent to MIR inequalities:

$$\frac{z_W - z_{MIR}}{z_W - z_{LP}}$$

- TIME, the total CPU time, in seconds, required process the instance.

Out of the 87 instances, three were infeasible (`ash608gpia-3col`, `enlight14`, `ns1766074`) and four (`acc-tight5`, `bnatt350`, `m100n500k4r1`, `neos-849702`) had z_{LP} equal to z_{OPT}. These instances were not considered. Ten instances exceeded our 60 hour CPU-time limit. Out of the remaining 70 instances, 42 instances presented $z_{MIR} = z_W$. Table 1 presents the performance indicators for the remaining 28 instances.

Table 1 Strength of wedge cuts versus MIR cuts alone

Instance	ORIG-GAP (%)	MIR-PERF (%)	W-PERF (%)	W-REL(%)	Time (s)
gmu-35-40.pre	0.01	0.07	9.94	99.26	122
eil33-2.pre	13.14	4.28	15.25	71.97	8972
neos-1337307.pre	0.4	3.76	6.45	41.66	32,572
opm2-z7-s2.pre	25.29	0.62	0.98	37.17	284,884
mik-250-1-100-1.pre	19.65	53.52	73.38	27.07	14
neos-686190.pre	23.7	4.61	5.54	16.82	33,697
mine-90-10.pre	11.15	12.4	14.51	14.6	970
cov1075.pre	14.29	3.6	4.19	13.9	95
mine-166-5.pre	45.09	6.57	7.58	13.35	1892
n3div36.pre	12.59	16.38	18.85	13.09	268,969
air04.pre	1.07	8.14	9.12	10.81	144,318
rococoC10-001000.pre	34.42	21.16	22.41	5.58	226
rmine6.pre	1.12	14.57	15.34	5	2751
reblock67.pre	11.61	21.38	22.46	4.81	446
ran16x16.pre	18.48	17.25	18.07	4.5	1
iis-bupa-cov.pre	26.4	1.22	1.26	3.59	9892
sp98ir.pre	1.37	4.63	4.77	2.88	10,242
iis-pima-cov.pre	19.33	2.1	2.14	1.94	34,347
iis-100-0-cov.pre	42.53	1.76	1.79	1.89	48
eilB101.pre	11.64	2.64	2.69	1.82	13,247
mzzv11.pre	4.86	26.99	27.11	0.43	49,301
roll3000.pre	13.9	21.83	21.91	0.37	356
dfn-gwin-UUM.pre	29.12	41.82	41.9	0.18	0
csched010.pre	18.52	3.89	3.9	0.15	647
msc98-ip.pre	1.56	17.78	17.81	0.14	26,752
neos-916792.pre	17.53	4.06	4.06	0.14	112
mcsched.pre	8.56	0.04	0.04	0.08	4634
beasleyC3.pre	68.44	15.58	15.59	0.05	0

It is well known that, when considering cuts from a single row of the simplex tableau, MIR cuts are very hard to outperform. Indeed, Fukasawa and Goycoolea [28] implemented an exact separator for *knapsack cuts*, a more general set of cuts that includes our wedge cuts, and tested it on the MIPLIB 3.0 and the MIPLIB 2003. Out of the 48 instances processed, on top of MIRs, knapsack cuts increased the gap closure by more than 1 percentage point for only 8 instances, and more than 5 percentage points for only one instance. It should be noted, however, that 44 instances could not be processed due to time constraints in that study.

In our experiment, we obtained noticeably better results. Out of the 70 instances processed, wedge cuts contributed to more than 1% of the gap closure for 20 instances, and more than 5% for 13 instances. In fact, for 5 instances, the contribution from

Table 2 Speed of wedge cuts versus MIR cuts

Instance	CUTS-MIR	CUTS-W	MIR-T	WEDGE-T	AVG-M
cov1075	582	174,970	0.16	0.20	13.60
eil33-2	30	566,411	7.61	8.35	32.63
gmu-35-40	27	58,555	0.83	1.16	56.85
mik-250-1-100-1	100	30,221	0.17	0.28	45.73
mine-166-5	1436	1,336,080	0.29	0.57	59.54
mine-90-10	1875	1,022,638	0.18	0.38	60.88
n3div36	48	3,838,798	32.06	41.67	45.83
neos-1337307	2263	8,302,981	1.13	1.52	39.20
neos-686190	254	3,162,782	5.56	5.54	26.98
opm2-z7-s2	7859	38,797,773	3.26	3.70	40.89

wedge cuts was greater than 25%. For two instances, `gmu-35-40` and `eil33-2`, the percentage was exceptionally high, at 99.26 and 71.97%, respectively. For the instance `gmu-35-40`, MIR cuts alone were only able to close 0.07% of the integrality gap, a negligible amount. The inclusion of wedge cuts improved that closure to 9.94%, which is noticeable. For the instance `mik-250-1-100-1`, although MIR cuts were able to reduce 53.52% of the gap, the inclusion of wedge cuts pushed that reduction to 73.38%, a significant improvement.

A side goal of our computational experiment to evaluate the efficiency of the enumeration algorithm presented in Sect. 3, with the trivial lifting algorithm of Sect. 5. In order to do that, we run the experiments again for the 10 instances for which wedge cuts presented the best performance, and we collected the additional statistics:

- CUTS-MIR and CUTS-W, the number of MIR cuts and wedge cuts, respectively, generated but not necessarily added to the relaxation,
- MIR-T and WEDGE-T, the average time needed to generate a single MIR cut and a single wedge cut, respectively, in milliseconds,
- AVG-M, the average number of times the inner loop of Algorithm 18 was repeated.

The results are presented on Table 2. On average, the time spent to generate one wedge cut was not much higher than the time spent to generate a single MIR cut. Note, however, that the number of wedge cuts generated, on all instances, was much larger than the number of MIR cuts, since we generate cuts for every tableau row, and for every integral non-basic variable. As a consequence, the total running time for many instances is prohibitively high, as seen in Table 1.

Next, instead of generating all possible wedge cuts for all possible combinations of rows and integral non-basic variables, we implement some heuristics to bring down the total number of cuts generated, and therefore the total running time of the algorithm. Our goal was to select a small subset of wedge cuts that, when added to the linear relaxation of the problem, yields approximately the same benefits, in terms of gap closure, as adding this entire family of cuts. We implemented three simple heuristics and tested their impact, both individually and combined, on the 10 instances for which wedge cuts presented the best performance.

Given a row and an integral non-basic variable, the first heuristic (DEPTH 5) limits the number wedge cuts generated to five. More precisely, we stop Algorithm 12 after five iterations, even if the stopping condition has not actually been reached. The motivation is that, as the vertices of the knapsack sets get farther away from f, the wedges become progressively thinner and, therefore, we would expect them to have progressively smaller practical impact. As shown in Table 3, this indeed turns out to be the case. After activating this heuristic, the gap closed is reduced by only 0.1 percentage points on average, a negligible amount, while the total running time is reduced by 7.4%. We conclude that this heuristic is moderately effective for most instances, although we do note that, for a few instances, the reduction on running time may not be worth the reduction in the gap closure (see instance `cov1075`, for example).

The second heuristic (100 ROWS) limits to 100 the number of tableau rows selected to generate wedge cuts. More precisely, after solving the initial linear relaxation, we discard all, but the 100 tableau rows that have the most fractional right-hand side. This is motivated by the fact that, if f is very close to an integral point, we would expect the number of generated wedges to be very small, and the generated cuts to be more numerically unstable. As shown in Table 3, this heuristic is very effective. While the gap closed is reduced by only 0.2 percentage points on average, the total running time is reduced by as much as 55%.

Given a tableau row, the third heuristic (100 RAYS) limits to 100 the number of integral non-basic variables selected to generated wedge cuts, based on their reduced cost. The motivation for this heuristic is that, when a non-basic integral variable is selected to play the role of s_1, it tends to receive better cut coefficients. It is natural, therefore, to try to assign better coefficients to the variables that have the most impact on the objective value. Unfortunately, as Table 3 shows, while this heuristic was very successful at bringing down the total running time (a 98% reduction, on average), it also had considerable impact on the gap closure (a reduction of 2.1 percentage points). We conclude that this heuristic is too aggressive for most instances. Unfortunately, we were unable to derive a better heuristic for selecting the integral non-basic variables.

Finally, we evaluated the impact of the three previous heuristics combined. The results are shown in Table 3, under the header COMBINED. As we see, while the three heuristics, together, were very effective at reducing the total running time of the cut generating procedure, they unfortunately also had considerable negative impact on the gap closure. Despite this, we note that wedge cuts still presented considerable improvement for a small subset of instances (see `gmu-35-40` and `mik-250-1-100-1`), under very reasonable running times.

7 Conclusion

In this paper, our main objective was to generate more generic one-row cuts; specifically, cuts that cannot be obtained via the lifting approach. Our strategy was to study cuts that are valid for a relaxation of the simplex tableau with one row and two integer

Table 3 Impact of cut selection heuristics

Instance	MIR-PERF	All combinations		Depth 5		100 rows		100 rays		Combined	
		W-PERF	W-TIME	W-PERF	W-TIME	W-PERF	W-TIME	W-PERF	W-TIME	W-PERF	W-TIME
cov1075.pre	3.6	4.19	95	3.83	96	3.81	11	3.79	46	3.79	10
eil33-2.pre	4.28	15.27	8972	15.27	8652	15.27	9061	5.7	136	5.69	212
gmu-35-40.pre	0.07	9.94	122	9.92	104	9.94	116	8.14	16	8.14	27
mik-250-1-100-1.pre	53.52	73.48	14	73.48	13	73.48	14	70.88	10	70.91	14
mine-166-5.pre	6.57	7.58	1892	7.58	1805	7.27	71	6.89	119	6.81	14
mine-90-10.pre	12.4	14.51	970	14.51	895	13.07	46	13.03	74	12.48	11
n3div36.pre	16.38	18.85	268,969	18.85	238,709	18.85	257,430	16.38	845	16.38	1218
neos-1337307.pre	3.76	6.45	32,572	6.45	31,355	6.44	1152	5.15	460	5.15	29
neos-686190.pre	4.61	5.51	33,697	5.51	32,521	5.39	10,524	5.3	491	5.21	286
opm2-z7-s2.pre	0.62	0.98	284,884	0.98	271,012	0.98	2717	0.7	7152	0.7	134
average	10.6	15.7	63,218.7	15.6	58,516.2	15.5	28,114.2	13.6	934.9	13.5	195.5

non-basic variables, using the framework of two-row cuts, as suggested by Conforti et al. [15]. By doing so, a two-row model with nice properties arises. We developed an algorithm to enumerate all the facet-defining inequalities for this model, which leads to an upper bound on its split rank, and we also developed a practical algorithm for solving the lifting problem that arises when additional integer non-basic variables are present. We implemented all the methods proposed, and performed computational experiments using real-world instances. Our cut generation scheme proved to be very fast in practice. As far as the effectiveness of the cuts is concerned, expectations were limited, since we generate a subset of knapsack cuts, which have been shown by Fukasawa and Goycoolea [28] to be only slightly stronger in practice than the MIR cuts they generalize. While, for most problems, wedge cuts did not seem to improve the integrality gap significantly when compared to MIR cuts alone, for some instances they did present a clear improvement, under very reasonable running times.

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A Trivial lifting function for S-free sets

In order to simplify the exposition, we examine here the lifting problem in a more general context and adopt the standard approach of the infinite relaxation, as well as its usual notation. We refer the reader to [16] for an introduction.

Let $S := \mathbb{Z}^m \cap Q$, where Q is some rational polyhedron. Let $f \in \mathbb{R}^m \setminus S$. We define

$$R_f := \left\{ y \in \mathbb{R}_+^{\mathbb{R}^m} \; : \; f + \sum_{r \in \mathbb{R}^m} r y_r \in S, \quad y \text{ has a finite support} \right\},$$

and a lifted version of R_f,

$$M_f := \left\{ y \in \mathbb{R}_+^{\mathbb{R}^m}, z \in \mathbb{Z}_+^{\mathbb{R}^m} \; : \; f + \sum_{r \in \mathbb{R}^m} r y_r + \sum_{r \in \mathbb{R}^m} r z_r \in S, \quad y, z \text{ have a finite support} \right\}.$$

We say that a function $\psi : \mathbb{R}^m \to \mathbb{R}$ is valid for R_f if $\sum_{r \in \mathbb{R}^m} \psi(r) y_r \geq 1$ for all $y \in R_f$. We say that $\psi, \pi : \mathbb{R}^m \to \mathbb{R}$ is valid for M_f if $\sum_{r \in \mathbb{R}^m} \psi(r) y_r + \sum_{r \in \mathbb{R}^m} \pi(r) z_r \geq 1$ for all $(y, z) \in M_f$. Given ψ valid for R_f, we say that π is a lifting of ψ if (ψ, π) is valid for M_f. For example, (ψ, ψ) is a lifting of ψ.

Proposition 19 *Let $\psi : \mathbb{R}^m \to \mathbb{R}$ be valid for R_f. For any $w : \mathbb{R}^m \to \mathbb{Z}^m \cap$ $\text{rec}(\text{conv}(S))$, the function $\pi(r) := \psi(r + w(r))$ is a lifting of ψ.*

Proof For all $(y, z) \in M_f$, we have

$$f + \sum_{r \in \mathbb{R}^m} r y_r + \sum_{r \in \mathbb{R}^m} r z_r \in S.$$

Since $z_r \geq 0$, $w(r) \in \mathbb{Z}^m$, and $w(r) \in \text{rec}(\text{conv}(S))$ for all $r \in \mathbb{R}^m$, we have $x + \sum_{r\in\mathbb{R}^m} w(r)z_r \in S$ for all $x \in S$. In particular,

$$f + \sum_{r\in\mathbb{R}^m} ry_r + \sum_{r\in\mathbb{R}^m} rz_r + \sum_{r\in\mathbb{R}^m} w(r)z_r \in S,$$

i.e.,

$$f + \sum_{r\in\mathbb{R}^m} ry_r + \sum_{r\in\mathbb{R}^m} (r + w(r))z_r \in S.$$

Because (ψ, ψ) is valid for M_f, we know that

$$\sum_{r\in\mathbb{R}^m} \psi(r)y_r + \sum_{r\in\mathbb{R}^m} \psi(r + w(r))z_r = \sum_{r\in\mathbb{R}^m} \psi(r)y_r + \sum_{r\in\mathbb{R}^m} \pi(r)z_r \geq 1,$$

for all $(y, z) \in M_f$. In other words, (ψ, π) is valid for M_f.

Corollary 20 *Let $\psi : \mathbb{R}^m \to \mathbb{R}$ be valid for R_f. Then,*

$$\pi(r) := \min_{w\in\mathbb{Z}^m\cap\text{rec}(\text{conv}(S))} \psi(r + w)$$

is a lifting of ψ.

Proposition 19 only gives sufficient conditions for π to be a lifting function. But if we insist on building π with a formula of the type $\pi(r) := \psi(r + w(r))$, then in all generality, it is necessary to have $w \in \mathbb{Z}^m \cap \text{rec}(\text{conv}(S))$. Proposition 21 shows that otherwise, we could construct M_f such that π is not a lifting.

Proposition 21 *Let $S := \mathbb{Z}^m \cap Q$, where Q is some rational polyhedron, and $w \notin \mathbb{Z}^m \cap \text{rec}(\text{conv}(S))$. There exist $f \in \mathbb{R}^m \setminus S$, $d \in \mathbb{R}^m$ and ψ valid for R_f such that if $\pi(d) = \psi(d + w)$, then π is not a lifting of ψ.*

Proof Since $w \notin \mathbb{Z}^m \cap \text{rec}(\text{conv}(S))$, there exists $\bar{x} \in S$ such that $\bar{x} + w \notin S$. Let $f := \bar{x} + w$. There exists $\varepsilon > 0$ such that $x \notin S$ for all $x \in \mathbb{R}^m$ such that $|x - f| \leq \varepsilon$. Let $\psi(r) := \frac{|r|}{\varepsilon}$. It is easy to verify that ψ is valid for R_f. We construct

$$\bar{y}_{-w} := 0, \quad \bar{z}_{-w} := 1, \quad \bar{y}_t := 0, \quad \bar{z}_t := 0, \text{ for all } t \neq -w.$$

Clearly, $f + \sum_{r\in\mathbb{R}^m} r\bar{y}_r + \sum_{r\in\mathbb{R}^m} r\bar{z}_r = \bar{x}$ so $(\bar{y}_r, \bar{x}_r) \in M_f$. However, we can let $d := -w$ and verify that

$$\sum_{r\in\mathbb{R}^m} \psi(r)\bar{y}_r + \sum_{r\in\mathbb{R}^m} \pi(r)\bar{z}_r = \pi(-w) = \psi(0) = 0 \not\geq 1,$$

showing that (ψ, π) is not valid for M_f.

Table 4 Directory structure and main files

Name	Description
build/	Directory that holds the compiled files
cmake/	Custom scripts used by CMake
googletest/	Google's C++ test framework
onerow/benchmark/	
src/	Source code for the benchmark code
instances/	MIPLIB 2010 instances
bases/	Optimal basis for the LP relaxation of each instance
output/	Logs and raw outputs produced by the benchmark code
tables/	Generated tables, in CSV format
onerow/library	
src/	Source code for the wedge-cut generator
include/	Header files for the library
tests/	Unit tests
qxx/	Auxiliary exact-arithmetic sparse-vector library
README.md	Instructions for compiling, running and producing the tables

B Source code

As previously mentioned, the source code for the wedge cut generator presented in Sect. 6 has been made available online [30] under the GNU General Public License, version 3. In this section, we give a brief overview of its structure and usage.

For clarity and reusability, the code has been split into two modules: the module `onerow/library` contains the code that generates wedge cuts from a given tableau row, as well helper functions that select and extract tableau rows from CPLEX. This module compiles to a shared library, which can be used by other applications. The module `onerow/benchmark`, on the other hand, contains the code that measures the performance of the library on instances from the MIPLIB, as well as the scripts that convert the raw outputs into the tables presented in Sect. 6. The structure of the project is presented in Table 4. Complete instructions to compile the source code, run the computational experiments and generate the tables have been included in the file `README.md`.

Next, we describe the main classes of the library module. Given a tableau row, wedge cuts can be generated by making use of the class `WedgeCutGenerator`, as follows:

```
Row r = ...;
WedgeCutGenerator wcg(r);
while(wcg.has_next()) {
    Constraint c = wcg.next();
}
```

A `Constraint` is composed by a list of rational coefficients and the right-hand side. A `Row` is a constraint with some extra information, such as reduced costs and the index of the basic variable. Although it is possible to construct a `Row` manually, the library comes with a helper class `CplexHelper` which is able to extract it from a CPLEX model:

```
CPXENVptr env = ...;
CPXLPptr lp = ...;
CplexHelper helper(env, lp);
Row *r = helper.get_tableau_row(1);
```

We recall that only rows with fractional right-hand side and integral basic variable are suitable for generating single-row cuts. The method `CplexHelper::find_good_rows` can be used for finding such rows. The method `CplexHelper::add_cut` simplifies adding the generated cut back into the CPLEX model. Before adding the cut, it verifies that the cut dynamism is within the allowed range, that it does cut off the current fractional solution and, optionally, that it does not cut off a given integral solution. Finally, instead of performing each of the previous steps in isolation, the helper class also includes the convenience method `CplexHelper::add_single_row_cuts`, which combines, in a single call, finding suitable rows, generating the cuts and adding them back into the model. In addition to `WedgeCutGenerator`, two other single-row cut generators, `GomoryCutGenerator` and `MIRCutGenerator`, have been implemented, and can be similarly used with `CplexHelper`. For more examples of usage, we refer to the unit tests included in the package, under `onerow/library/tests`.

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