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FULL LENGTH PAPER

On the impact of running intersection inequalities for globally solving polynomial optimization problems

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Abstract
We consider global optimization of nonconvex problems whose factorable reformulations contain a collection of multilinear equations of the form $z_e = \prod_{v \in e} z_v$, $e \in E$, where E denotes a set of subsets of cardinality at least two of a ground set. Important special cases include multilinear and polynomial optimization problems. The multilinear polytope is the convex hull of the set of binary points z satisfying the system of multilinear equations given above. Recently Del Pia and Khajavirad introduced running intersection inequalities, a family of facet-defining inequalities for the multilinear polytope. In this paper we address the separation problem for this class of inequalities. We first prove that separating flower inequalities, a subclass of running intersection inequalities, is NP-hard. Subsequently, for multilinear polytopes of fixed degree, we devise an efficient polynomial-time algorithm for separating running intersection inequalities and embed the proposed cutting-plane generation scheme at every node of the branch-and-reduce global solver `BARON`. To evaluate the effectiveness of the proposed method we consider two test sets: randomly generated multilinear and polynomial optimization problems of degree three and four, and computer vision instances from an image restoration problem Results show that running intersection cuts significantly improve the performance of `BARON` and lead to an average CPU time reduction of 50% for the random test set and of 63% for the image restoration test set.

Keywords Branch-and-cut · Polynomial optimization · Running-intersection inequalities · Multilinear polytope · Separation algorithm · Mixed-integer nonlinear optimization

Mathematics Subject Classification 90C11 · 90C26 · 90C57

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1 Introduction

Constructing strong and cheap to compute convex relaxations is key to the efficiency of global optimization algorithms. Factorable programming is a widely used methodology in global optimization of mixed-integer nonlinear optimization problems (MINLPs) for bounding general nonconvex functions [24]. These techniques iteratively decompose a factorable function, through the introduction of auxiliary variables and constraints for intermediate nonlinear expressions, until each intermediate expression can be convexified effectively. General-purpose global solvers such as `BARON` [22], `COUENNE` [5], `SCIP` [6], `LindoGlobal` [23], and `ANTIGONE` [27], rely on factorable programming bounds. Hence, improving the quality of these relaxations has an immediate impact on our ability to solve nonconvex problems to global optimality.

Multilinear sets and polytopes Factorable reformulations of many types of MINLPs, such as mixed-integer polynomial optimization problems, contain a collection of multilinear equations of the form $z_e = \prod_{v \in e} z_v$, $e \in E$, where E denotes a set of subsets of cardinality at least two of a ground set V. Let us define the set of points satisfying all multilinear equations present in a factorable reformulation of a MINLP as

$$\tilde{\mathcal{S}} = \left\{ z \in \mathbb{R}^{V+E} : z_e = \prod_{v \in e} z_v \ \forall e \in E, z_v \in [0,1] \ \forall v \in V_1, \ z_v \in \{0,1\} \ \forall v \in V_2 \right\}, \tag{1}$$

where V_1 and V_2 form a partition of V and are the index sets corresponding to continuous and binary variables, respectively. Notice that, if the continuous variables are defined over a general rectangular region, then one can obtain a set of the form $\tilde{\mathcal{S}}$ after some scaling and shifting of variables. It then follows that the convex hull of $\tilde{\mathcal{S}}$ is a polytope and the projection of its vertices onto the space of the variables z_v, $v \in V$, is given by $\{0,1\}^V$ (see for example [37]). Consequently, the facial structure of the convex hull of $\tilde{\mathcal{S}}$ can be equivalently studied by considering the following binary set:

$$\left\{ z \in \{0,1\}^{V+E} : z_e = \prod_{v \in e} z_v \ \forall e \in E \right\}. \tag{2}$$

There is a one-to-one correspondence between sets of form (2) and hypergraphs $G = (V, E)$ [12]. Henceforth we refer to (2) as the *multilinear set* of G and denote it by $\mathcal{S}_G$, and refer to its convex hull as the *multilinear polytope* of G and denote it by MP_G. If all multilinear equations defining $\mathcal{S}_G$ are bilinear, the multilinear polytope coincides with the Boolean quadric polytope defined by Padberg [28] in the context of $0-1$ quadratic optimization. Over the past three decades, a significant amount of research has been devoted to studying the facial structure of the Boolean quadric polytope and these theoretical findings have had a significant impact on the performance of branch-and-cut based algorithms for mixed-integer quadratic optimization problems (MIQCPs) [3,8,19,43]. In contrast, as we detail next, for higher degree multilinear polytopes, similar polyhedral studies are rather scarce.

The explicit characterization of the multilinear polytope of the complete hypergraph, that is, the hypergraph whose edge set consists of all subsets of nodes of cardinality at least two, has been presented in several studies (see for example [35]). In [12] the authors study the facial structure of the multilinear polytope of general hypergraphs and develop the theory of a number of lifting operations, giving rise to various types of facet-defining inequalities. In [13], the authors study the decomposability properties of the multilinear polytope. Namely, they derive necessary and sufficient conditions under which we have $\mathrm{MP}_G = \bigcap_{j \in J} \mathrm{MP}_{G_j}$, where each hypergraph G_j has a much simpler structure than the original hypergraph G. In [14,15], the authors study the complexity of the facet-description of the multilinear polytope in conjunction with the acyclicity degree of the underlying hypergraph. Explicit characterizations of the multilinear polytope of Berge-acyclic and γ-acyclic hypergraphs are provided in [14]. Moreover, as a byproduct, [14] introduces flower inequalities, a family of valid inequalities for the multilinear polytope, which generalize 2-link inequalities defined in [11]. Subsequently, in [15], the authors introduce running intersection inequalities, a significant generalization of flower inequalities and identify sufficient conditions under which the proposed inequalities are facet-defining. Moreover, they show that the polytope obtained by adding all running intersection inequalities to the standard linearization of the multilinear set coincides with the multilinear polytope of a large subclass of β-acyclic hypergraphs.

Quadratization approaches In order to capitalize on existing algorithms for $0-1$ quadratic optimization, various techniques have been developed to reduce higher degree binary polynomial optimization problems to quadratic ones, at the cost of an increase in the number of variables. These methods are often referred to as quadratization approaches. In [1], the authors perform a systematic study of the existing quadratization techniques. They provide tight lower and upper bounds on the number of added variables in the worst case. In particular, they show that, for each fixed degree d, there are polynomials of n variables and of degree d for which every quadratization must involve at least $\Omega(n^{d/2})$ added variables. In [17], the authors demonstrate the usefulness of quadratization in some computer vision applications. In [9], the authors propose a different type of quadratization scheme that is most effective when the original multilinear set is reducible; that is, every set $e \in E$ is a union of two other sets in E. Otherwise, auxiliary variables are added to the model to make $\mathcal{S}_G$ reducible.

Convex and concave envelopes of multilinear functions There has been a stream of research in the global optimization literature on characterizing the convex/concave envelope of a multilinear function defined as $\sum_{e \in E} c_e \prod_{v \in e} z_v$, where $c_e \in \mathbb{R} \backslash \{0\}$ for all $e \in E$ and $z_v \in [l_v, u_v]$ such that $-\infty < l_v < u_v < \infty$ for all $v \in V$. Since multilinear functions are closed under scaling and shifting of variables, without loss of generality one can assume that $l_v = 0$ and $u_v = 1$ for all $v \in V$. It then follows that an extended formulation for the convex hull of a multilinear function (and hence its convex and concave envelopes) is given by the image of the multilinear polytope MP_G under the linear mapping $(z_v, z_e) \rightarrow (z_v, \sum_{e \in E} c_e z_e)$. For a few classes of structured multilinear functions, explicit characterizations of the envelopes are available. These results address trilinear terms over a box [25,26], special forms of bilinear functions over the unit hypercube [30], special forms of multilinear functions over the unit hypercube and some discrete sets [34], bilinear and polynomial covering sets with certain sign restrictions and no upper bounds on variables [38], and submodular functions over various polyhedral subdivisions of a box [39].

For bounding a general multilinear function, however, a common practice is to utilize a termwise scheme in which each multilinear term is relaxed by a recursive application of bilinear envelopes [24,31,40]. This termwise factorable scheme is simple to implement and has been deployed in all general-purpose global solvers; however, in general, it leads to very poor bounds. As a notable exception, in [2], the authors propose a cutting plane generation framework in which certain facets corresponding to the convex hull of a multilinear function are generated by solving a linear optimization problem (LP) and are added at every node of the global solver BARON. However, the size of this LP grows exponentially with the number of variables in the multilinear function. To control the size of the LP, the authors present a decomposition algorithm to break down a multilinear function into a collection of lower-dimensional multilinear functions. In addition, they devise a customized simplex algorithm for solving the separation problem that outperforms the state-of-the-art LP solvers by several orders of magnitude. Their computational study on various sets of multilinear and polynomial optimization problems show that the proposed cuts lead to significant reductions in CPU time and enable BARON to solve many more problems to global optimality.

Our contribution In this paper, we study the separation problem for running intersection inequalities [15], a family of valid inequalities for the multilinear polytope. In [14] the authors consider flower inequalities, a subclass of running intersection inequalities, and show that, over γ-acyclic hypergraphs, the separation problem can be solved in strongly polynomial-time; i.e., in a number of iterations bounded by a polynomial in $|V|$ and $|E|$. They also mention that, over balanced hypergraphs, a generalization of γ-acyclic hypergraphs, separation of flower inequalities can be done in polynomial-time by solving an LP. However, we are not aware of any application in which the corresponding hypergraph is balanced. In this paper, we prove that the separation problem for flower inequalities over general hypergraphs is NP-hard. Subsequently, we consider hypergraphs with a fixed rank, where the rank r of G is defined as the maximum cardinality of an edge in E. This assumption is based on the observation that, for the multilinear sets that appear in MINLPs, we often have $r \ll |V|$. In fact, for all practical purposes we can assume that $r \leq 5$ and therefore it is reasonable to assume that r is a fixed parameter. We then show that if G has a fixed rank, the separation problem for running intersection inequalities can be solved in $O(|E|^2)$ operations.

To exploit the local information regarding bounds on variables for cut generation, as well as to utilize running intersection inequalities for local feasibility-based and optimality-based range reductions, we devise an efficient implementation of the separation algorithm within a branch-and-cut framework. Our proposed cut generation algorithm is embedded at every node of BARON's branch-and-reduce algorithm. Extensive computational results are presented for globally solving multilinear and polynomial optimization problems of degree three and four. We consider two types of test problems. The first one consists of randomly generated instances taken from [2], while the second one contains computer vision instances from an image restoration problem [11]. Results for the test problems show that the incorporation of running intersection cuts in BARON reduces the average CPU time and number of nodes in the search tree by 50% and 75% for the random test set, and by 63% and 42% for the image restoration test set.

Organization In Sect. 2, we provide a brief overview of running intersection and flower inequalities. In Sect. 3, we show that the separation problem for flower inequalities over general hypergraphs is NP-hard. Subsequently, in Sect. 4, we consider fixed-rank hypergraphs and present an efficient polynomial-time separation algorithm for running intersection inequalities. With the objective of embedding the proposed separation algorithm in a branch-and-cut framework, in Sect. 5, we present an enhanced implementation that conducts most of the expensive computations at the root node. Finally, computational results on a variety of multilinear and polynomial optimization problems are presented in Sect. 6.

2 Running intersection inequalities

In [15] the authors introduce running intersection inequalities, a class of valid inequalities for the multilinear polytope. For a self-contained exposition, in the following we briefly review some of the key concepts related to these inequalities which will be used for the subsequent developments.

To define running intersection inequalities, we make use of the notion of running intersection property. This concept was first introduced in the database community to study acyclic databases [4] and has close connections to the notion of tree-width in intersection graphs which has been used, for example in [7], to study the complexity of polynomial optimization problems. A set F of subsets of a finite set V has the *running intersection property* if there exists an ordering $p_1, p_2, \ldots, p_m$ of the sets in F such that

$$\text{for each } k = 2, \ldots, m, \text{ there exists } j < k \text{ such that } p_k \cap \left(\bigcup_{i<k} p_i\right) \subseteq p_j. \tag{3}$$

Throughout the paper, we refer to an ordering $p_1, p_2, \ldots, p_m$ satisfying (3) as a *running intersection ordering* of F. Each running intersection ordering $p_1, p_2, \ldots, p_m$ of F induces a collection of sets

$$N(p_1) := \emptyset, \qquad N(p_k) := p_k \cap \left(\bigcup_{i<k} p_i\right) \text{ for } k = 2, \ldots, m. \tag{4}$$

Definition 1 Consider a hypergraph $G = (V, E)$. Let $e_0 \in E$ and let e_k, $k \in K$, be a collection of edges in E satisfying the following conditions:

(i) $|e_0 \cap e_k| \geq 2$ for all $k \in K$,
(ii) $e_0 \cap e_k \not\subseteq e_0 \cap e_{k'}$ for any $k, k' \in K$,
(iii) the set $\tilde{E} := \{e_0 \cap e_k : k \in K\}$ has the running intersection property.

Consider a running intersection ordering of $\tilde{E}$ with the corresponding sets $N(e_0 \cap e_k)$, for $k \in K$, as defined in (4). For each $k \in K$ with $N(e_0 \cap e_k) \neq \emptyset$, let w_k be a node in $N(e_0 \cap e_k)$. We define a *running intersection inequality* as

$$-\sum_{k \in K : N(e_0 \cap e_k) \neq \emptyset} z_{w_k} + \sum_{v \in e_0 \setminus \bigcup_{k \in K} e_k} z_v + \sum_{k \in K} z_{e_k} - z_{e_0} \leq \omega - 1, \tag{5}$$

where

$$\omega = \left| e_0 \setminus \bigcup_{k \in K} e_k \right| + \left| k \in K : N(e_0 \cap e_k) = \emptyset \right|.$$

We refer to e_0 as the *center* and to e_k, $k \in K$, as the *neighbors*.

The above definition for running intersection inequalities depends on a running intersection ordering of the set $\tilde{E}$ defined in condition (iii). It can be shown that such an ordering is, in general, not unique. However, in [12], the authors prove that the system of all running intersection inequalities centered at e_0 with neighbors e_k, $k \in K$, is independent of the running intersection ordering. Moreover, they present conditions under which running intersection inequalities define facets of the multilinear polytope MP_G.

In the special case where $N(e_0 \cap e_k) = \emptyset$ for all $k \in K$; i.e., $e_0 \cap e_k \cap e_{k'} = \emptyset$ for all distinct $k, k' \in K$, running intersection inequalities simplify to flower inequalities introduced in [14]. The *flower inequality* centered at e_0 with the neighbors e_k, $k \in K$ is given by:

$$\sum_{v \in e_0 \setminus \bigcup_{k \in K} e_k} z_v + \sum_{k \in K} z_{e_k} - z_{e_0} \leq \left| e_0 \setminus \bigcup_{k \in K} e_k \right| + |K| - 1. \tag{6}$$

In this paper, we are interested in utilizing running intersection inequalities in a branch-and-cut framework. As we detailed in Sect. 1, given a collection of multilinear terms corresponding to a factorable reformulation of a MINLP, one can obtain a set of the form (1) after possibly scaling and shifting some of the continuous variables. In a branch-and-cut framework, the variables bounds change in different parts of the tree. Clearly, running intersection cuts corresponding to the variable bounds at the root node remain valid throughout the search tree. However, for cut generation, it is highly beneficial to exploit tighter variable bounds at each node of the tree. To this end, at each node, one can construct the set $\tilde{\mathcal{S}}$ defined by (1) using local bounds on variables and generate cutting planes accordingly. However, such an approach is fairly expensive as the structure of the hypergraph may change as a result of each branching or range reduction. Hence, to address this trade-off, in this paper, we pursue a different alternative. Define

$$\mathcal{S}^b_G = \left\{ z \in \mathbb{R}^{V+E} : z_e = \prod_{v \in e} z_v \; \forall e \in E, \; z_v \in [l_v, u_v], \forall v \in V \right\},$$

where $0 \leq l_v < u_v < \infty$ for all $v \in V$. Define $\tilde{z}_v = z_v/u_v$ for all $v \in V$ and $\tilde{z}_e = \prod_{v \in e} \tilde{z}_v$ for all $e \in E$. Let $u_e = \prod_{v \in e} u_v$. It then follows that $\tilde{z}_e = z_e/u_e$ for all $e \in E$. Hence, the following inequality is valid for $\mathcal{S}^b_G$:

$$-\sum_{k \in K: N(e_0 \cap e_k) \neq \emptyset} \frac{z_{w_k}}{u_{w_k}} + \sum_{v \in e_0 \setminus \bigcup_{k \in K} e_k} \frac{z_v}{u_v} + \sum_{k \in K} \frac{z_{e_k}}{u_{e_k}} - \frac{z_{e_0}}{u_{e_0}} \leq \omega - 1, \qquad (7)$$

where ω is as defined in (5).

3 Separation of flower inequalities over general hypergraphs

In [14] the authors consider flower inequalities as defined in (6) and show that, over γ-acyclic hypergraphs, the separation problem can be solved in strongly polynomial-time. They also mention that over balanced hypergraphs, a generalization of γ-acyclic hypergraphs, separation of flower inequalities can be done in polynomial-time by solving an LP. In this section, we prove that separating flower inequalities over general hypergraphs is NP-hard. First, we formally define the separation problem for a general family of inequalities (see [33] for more details).

The separation problem Given a hypergraph $G = (V, E)$, a family of inequalities valid for MP_G, and a vector $\bar{z} \in [0, 1]^{V+E}$, decide whether $\bar{z}$ satisfies all the inequalities in the family or not, and in the latter case, find an inequality in the family that is violated by $\bar{z}$.

Before stating our main result, let us recall a widely-used polyhedral relaxation of $\mathcal{S}_G$ that is obtained by replacing each multilinear term $z_e = \prod_{v \in e} z_v$, by its convex hull over the unit hypercube:

$$\begin{aligned} \text{MP}^{\text{LP}}_G = \Big\{ & z : z_v \leq 1, \ \forall v \in V, \\ & z_e \geq 0, \ z_e \geq \sum_{v \in e} z_v - |e| + 1, \ \forall e \in E, \\ & z_e \leq z_v, \forall v \in e, \ \forall e \in E \Big\}. \end{aligned} \qquad (8)$$

The above relaxation is often referred to as the *standard linearization* of the multilinear set [10]. For a rank-r hypergraph $G = (V, E)$, the system defining MP^{LP}_G has at most $|V| + r|E|$ inequalities. Hence, for our computational complexity analysis, it is reasonable to assume that the point $\bar{z}$ is present in MP^{LP}_G, as otherwise, one can easily identify an inequality in system (8) that is violated by $\bar{z}$.

To prove the next theorem, we need to give some definitions. A *matching* in a hypergraph $G = (V, E)$ is a subset M of E with the property that $e \cap f = \emptyset$ for all $e, f \in M$ with $e \neq f$. A matching in G is called *perfect* if each node in V is contained in exactly one edge of the matching. We say that a hypergraph G is *3-uniform* if each edge contains exactly three nodes.

Theorem 1 *Given a hypergraph $\bar{G}$ and a vector $\bar{z} \in MP^{LP}_{\bar{G}}$, the following problems are NP-hard:*

(P1) *Is there a flower inequality for $\bar{G}$ centered at an edge e_0 of $E(\bar{G})$ violated by $\bar{z}$?*
(P2) *Is there a flower inequality for $\bar{G}$ violated by $\bar{z}$?*

Proof In order to show that Problems (P1) and (P2) are NP-hard, we give polynomial reductions from the 3-Dimensional Matching Problem (3-DMP). We refer the reader to [21] for the definition and NP-hardness of the 3-DMP. The 3-DMP is a special case of the following problem, which is therefore NP-hard as well.

(P3) Given a 3-uniform hypergraph $G = (V, E)$, is there a matching M of G with $|M| = \frac{|V|}{3}$?

Let $G = (V, E)$ be a 3-uniform hypergraph with a number of nodes divisible by 3 and let $n := |V|$.

Reduction to perfect matching with costs We define the set of loops $L := \{\{v\} : v \in V\}$, and let $\tilde{G} := (V, E \cup L)$. In the following, for ease of notation, we often identify a loop $\{v\}$ with the node v. Let $\alpha := \frac{3}{n+1}$. Since $n \geq 3$, α satisfies $0 < \alpha < 1$. We define the *cost* of each element of $E(\tilde{G})$ as

$$w_e := \alpha \quad \forall e \in E$$
$$w_v := \alpha \quad \forall v \in L.$$

Accordingly, we define the *cost* of each subset $\tilde{M}$ of $E \cup L$ as

$$c(\tilde{M}) := \sum_{v \in \tilde{M} \cap L} c_v + \sum_{e \in \tilde{M} \cap E} c_e = \alpha |\tilde{M}|.$$

We now show that Problem (P3) is equivalent to the following problem:

(P4) Is there a perfect matching $\tilde{M}$ of $\tilde{G}$ of cost $c(\tilde{M}) < 1$?

There is a natural bijection between matchings in G and perfect matchings in $\tilde{G}$. Namely, given a perfect matching $\tilde{M}$ of $\tilde{G}$, the set M obtained from $\tilde{M}$ by deleting all loops is a matching of G. Conversely, given a matching M of G, the set $\tilde{M} := M \cup \{\{v\} : v \notin \bigcup_{e \in M} e\}$ is a perfect matching of $\tilde{G}$. Since all edges of G contain three nodes, for a matching M of G and its corresponding perfect matching $\tilde{M}$ of $\tilde{G}$ we have

$$|\tilde{M}| = |M| + (n - 3|M|) = n - 2|M|.$$

Hence,

$$c(\tilde{M}) = \alpha |\tilde{M}| = \alpha(n - 2|M|).$$

This implies that $|M| = \frac{n}{3}$ if and only if $c(\tilde{M}) = \alpha(n - 2\frac{n}{3}) = \frac{n}{n+1}$, where we used the definition of α. Now consider a matching M of G that satisfies $|M| < \frac{n}{3}$. Then $|M| \leq \frac{n-3}{3}$ and this happens if and only if $c(\tilde{M}) \geq \alpha(n - 2\frac{n-3}{3}) = \frac{n+6}{n+1}$. This shows that there is a matching M of G with $|M| = \frac{n}{3}$ if and only if there is a perfect matching $\tilde{M}$ of $\tilde{G}$ of cost $c(\tilde{M}) < 1$.

Reduction to problem (P1) Let $\bar{G} := (V, E \cup e_0)$, where $e_0 := V$. We define the vector $\bar{z} \in [0, 1]^{V \cup E(\bar{G})}$ as follows:

$$\begin{aligned} \bar{z}_{e_0} &:= 0 \\ \bar{z}_e &:= 1 - \alpha \qquad \forall e \in E \\ \bar{z}_v &:= 1 - \alpha \qquad \forall v \in V. \end{aligned}$$

First, we show that the vector $\bar{z}$ is in the standard linearizaton $\text{MP}^{\text{LP}}_{\bar{G}}$. Using the definition of $\bar{z}$, we can rewrite the inequalities in the definition of $\text{MP}^{\text{LP}}_{\bar{G}}$ with $z = \bar{z}$. The inequalities corresponding to the nodes of $\bar{G}$ are:

$$\alpha \geq 0 \quad \forall v \in V. \tag{9}$$

The inequalities corresponding to an edge $e \in E$ are:

$$\alpha \leq 1 \qquad \forall e \in E \tag{10}$$

$$\alpha \leq \sum_{v \in e} \alpha \qquad \forall e \in E \tag{11}$$

$$\alpha \geq \alpha \qquad \forall v \in e, \qquad \forall e \in E. \tag{12}$$

Finally, the inequalities corresponding to edge e_0 are:

$$1 \leq 1 \tag{13}$$

$$1 \leq \sum_{v \in V} \alpha \tag{14}$$

$$1 \geq \alpha \qquad \forall v \in V. \tag{15}$$

We now show that all inequalities (9)–(15) are satisfied. Clearly, inequalities (12) and (13) are satisfied. Since $0 < \alpha < 1$, it follows that (9), (10), and (15) are also satisfied. Since each edge $e \in E$ contains three nodes, we have $\sum_{v \in e} \alpha = 3\alpha > \alpha$, implying (11) is satisfied. Finally, $\sum_{v \in V} \alpha = n\alpha = \frac{3n}{n+1} > 1$, and thus (14) also is satisfied.

We now prove that Problem (P4) is equivalent to Problem (P1) with the defined $\bar{G}$, e_0, and $\bar{z}$. To this end, we show that there exists a flower inequality for $\bar{G}$ centered at e_0 violated by $\bar{z}$ if and only if there is a perfect matching $\tilde{M}$ of $\tilde{G}$ of cost $c(\tilde{M}) < 1$.

Every edge $e \in E$ is adjacent to e_0 and satisfies $|e_0 \cap e| = 3 \geq 2$. Therefore, there exists a flower inequality for $\bar{G}$ centered at e_0 violated by $\bar{z}$ if and only if there exist a nonempty collection of edges e_k, $k \in K$, in E with $e_k \cap e_{k'} = \emptyset$ for all distinct $k, k' \in K$, such that

$$\sum_{v \in e_0 \setminus \bigcup_{k \in K} e_k} \bar{z}_v + \sum_{k \in K} \bar{z}_{e_k} - \bar{z}_{e_0} > \left| e_0 \setminus \bigcup_{k \in K} e_k \right| + |K| - 1,$$

or, equivalently,

$$\sum_{v \in e_0 \setminus \bigcup_{k \in K} e_k} (1 - \bar{z}_v) + \sum_{k \in K} (1 - \bar{z}_{e_k}) < 1 - \bar{z}_{e_0}.$$

Using the definition of $\bar{z}$ and c, the latter reduces to

$$\sum_{v \in e_0 \setminus \bigcup_{e \in K} e} c_v + \sum_{e \in K} c_e < 1. \tag{16}$$

Assume now that there exists a nonempty collection of edges e_k, $k \in K$, with $e_k \cap e_{k'} = \emptyset$ for all distinct $k, k' \in K$, that satisfies (16). Define $\tilde{M} := K \cup \{\{v\} : v \notin \bigcup_{k \in K} e_k\}$. Then $\tilde{M}$ is a perfect matching of $\tilde{G}$ with $c(\tilde{M}) < 1$. Conversely, assume that $\tilde{M}$ is a perfect matching of $\tilde{G}$ with $c(\tilde{M}) < 1$. Then, the set e_k, $k \in K$, obtained from $\tilde{M}$ by deleting all loops is a nonempty collection of edges with $e_k \cap e_{k'} = \emptyset$ for all distinct $k, k' \in K$, that satisfies (16).

This completes the proof that Problem (P1) is NP-hard.

Reduction to Problem (P2) Consider a flower inequality for $\bar{G}$ centered at an edge $e \in E$. We show that this flower inequality is always valid at $z = \bar{z}$. Note that we must have $|K| = 1$, thus we denote by k the only index in K. There are two possibilities: either e_k is an edge in E with $|e \cap e_k| = 2$, or $e_k = e_0$. In the first case, the corresponding flower inequality is

$$z_v + z_{e_k} - z_e \leq 1,$$

where v is the unique node in $e \setminus e_k$. Such an inequality is always satisfied by $\bar{z}$ because $\bar{z}_v = \bar{z}_{e_k} = \bar{z}_e = \alpha$, and $\alpha < 1$. In the second case, the corresponding flower inequality is

$$z_{e_0} - z_e \leq 0.$$

This inequality is always satisfied at $\bar{z}$ since $\bar{z}_{e_0} = 0$, $\bar{z}_e = \alpha$, and $\alpha > 0$.

This implies that there exists a flower inequality for $\bar{G}$ violated by $\bar{z}$ if and only if there exists a flower inequality for $\bar{G}$ centered at e_0 violated by $\bar{z}$. This shows that Problem (P2) is NP-hard as well. $\square$

4 Separation of running intersection inequalities over fixed-rank hypergraphs

The NP-hardness results of Sect. 3 are based on the assumption that, given a multilinear set $\mathcal{S}_G$, the rank r of the hypergraph $G = (V, E)$ is a problem input. However, for multilinear sets that appear in MINLPs we often have $r \ll |V|$. More precisely, in a polynomial optimization problem, $|V|$ corresponds to the number of variables, r corresponds to the highest degree of the polynomial involved, and $|E|$ corresponds to the number of distinct monomial terms. In most nontrivial MINLPs that appear in applications we have at least a few hundreds of variables and many more multilinear terms, while the degree of the multilinear set is often quite low. Indeed, we often have $r \leq 5$. See for example test libraries in [32]. That is, for these problems, we have $r \ll |V|$ and $|E| \gg 2^r$. Therefore it is reasonable to assume that r is a fixed parameter. In this section, we consider the separation problem over all running intersection inequalities. As we detail in the following, if r is fixed, this separation problem can be solved in strongly polynomial-time, i.e., in a number of iterations bounded by a polynomial in $|V|$ and $|E|$.

For a general hypergraph $G = (V, E)$, the number of all running intersection inequalities is often very large. In the following, we a describe a preprocessing step for our separation algorithm that enables us to solve the separation problem over a small subset of running intersection inequalities. In the sequel, we refer to such inequalities as the *separation system*. As we detail later, this step is an essential part of our separation algorithm as it significantly reduces both time and memory requirements. Let $\bar{z} \in [0, 1]^{V+E}$. Let e_0 denote an edge in E, and denote by E_{e_0} the subset of E containing, for every $f \subseteq e_0$ with $|f| \geq 2$, among all the edges $e \in E \setminus \{e_0\}$ with $e_0 \cap e = f$, only one that maximizes $\bar{z}_e$. The set $\mathcal{I}_{e_0}$ is then defined as the set of running intersection inequalities with center e_0, and neighbors $e_k \in E_{e_0}$, for $k \in K$, such that for each $k \in K$, the node w_k is chosen as one in $N(e_0 \cap e_k)$ that minimizes $\bar{z}_{w_k}$. Finally, the separation system $\mathcal{I}$ is defined as $\mathcal{I} := \bigcup_{e_0 \in E} \mathcal{I}_{e_0}$. Given a running intersection inequality (5) and a vector $\bar{z} \in [0, 1]^{V+E}$, we define the *violation of the inequality by* $\bar{z}$ as the difference between the left-hand side of (5) evaluated in $\bar{z}$ and its right-hand side. The following proposition establishes the key property of running intersection inequalities that are present in the separation system.

Proposition 1 *Let $G = (V, E)$ be a hypergraph and let $\bar{z} \in [0, 1]^{V+E}$. Consider a running intersection inequality centered at e_0 with neighbors e_k, $k \in K$. Then $\mathcal{I}_{e_0}$ contains a running intersection inequality centered at e_0 with neighbors e'_k, $k \in K$, such that $e_0 \cap e_k = e_0 \cap e'_k$ for all $k \in K$ whose violation by $\bar{z}$ is at least as large as the violation of the original inequality by $\bar{z}$.*

Proof Consider a running intersection inequality (5) centered at e_0 with neighbors e_k, $k \in K$. For each $k \in K$ with $N(e_0 \cap e_k) \neq \emptyset$, let w_k be the node selected from $N(e_0 \cap e_k)$ to construct this inequality. For each $k \in K$, we define the new neighbor e'_k as an edge e that maximizes $\bar{z}_e$ among all edges in $e \in E$ with $e_0 \cap e = e_0 \cap e_k$. By construction, such an edge e'_k is always present in E_{e_0}. For each $k \in K$ with $N(e_0 \cap e_k) \neq \emptyset$, we define the node u'_k as a node u in $N(e_0 \cap e_k)$ that minimizes $\bar{z}_u$.

We now obtain a new running intersection inequality from the original one by replacing each neighbor e_k with the corresponding e'_k and by replacing each node w_k with the corresponding u'_k. The new inequality is a running intersection inequality as for each edge $k \in K$ we have $e_0 \cap e_k = e_0 \cap e'_k$, implying $N(e_0 \cap e_k) = N(e_0 \cap e'_k)$. Moreover, from the definition of the separation system it follows that this inequality is present in $\mathcal{I}_{e_0}$. Clearly, the left-hand side of the new inequality evaluated at $\bar{z}$ is not smaller than the left-hand side of the original running intersection inequality evaluated at $\bar{z}$. Since the two inequalities have the same right-hand side, it follows that the violation of the new inequality by $\bar{z}$ is at least as large as the violation of the original inequality by $\bar{z}$. $\square$

Proposition 1 implies that the separation problem over all running intersection inequalities can be equivalently solved over the separation system $\mathcal{I}$. More precisely, if there exists a running intersection inequality that is violated by $\bar{z}$ in the original system, then there also exists an inequality in the separation system that is violated by $\bar{z}$. We say that a running intersection inequality is *maximally violated by* $\bar{z}$ if the violation of the inequality by $\bar{z}$ is maximum among all running intersection inequalities. Then, Proposition 1 implies that the separation system contains a running intersection inequality that is maximally violated by $\bar{z}$. As we detail later, the computational cost of the separation problem over all running intersection inequalities depends on the maximum number of neighbours present in running intersection inequalities. Hence, to control the cost of the separation algorithm, we now consider the problem of separating $\bar{z}$ over all running intersection inequalities with at most q neighbors. We refer to this restricted separation problem as *the q-separation problem.*

From Proposition 1, it follows that the q-separation problem can be equivalently solved over inequalities with at most q neighbors that are present the separation system. We refer to the set of all such inequalities as the q-separation system. Throughout this section, we assume that a rank-r hypergraph $G = (V, E)$ is represented by an incidence-list in which edges are stored as objects, and every edge stores the vertices it contains. In order to use efficient search algorithms, we assume that the vertex list for each edge is sorted. Otherwise, such a sorted data structure can be obtained in $O(r|E|)$ operations by using some integer sorting algorithm such as counting sort.

Proposition 2 *Given a rank-r hypergraph $G = (V, E)$ and a vector $\bar{z} \in [0, 1]^{V+E}$, there exists an algorithm that solves the q-separation problem in $O(|E|(r2^r|E| + 2^{rq}q^2r))$ operations.*

Proof Let e_0 be an edge in E. We show how to solve the separation problem over all inequalities in the q-separation system that are centered at e_0. By applying the algorithm $|E|$ times, we can then solve the q-separation problem.

Let the sets $\mathcal{I}_{e_0}$ and E_{e_0} be as defined before. Denote by $\mathcal{I}^q_{e_0}$ the set of all inequalities in $\mathcal{I}_{e_0}$ with at most q neighbors. By Proposition 1, to solve the q-separation problem, it suffices to consider inequalities in $\mathcal{I}^q_{e_0}$. In our separation algorithm, we explicitly generate all running intersection inequalities in $\mathcal{I}^q_{e_0}$. First we construct the set E_{e_0}. For a rank-r hypergraph G, the number of edges in E_{e_0} is at most $2^r - r - 1$. This follows from condition (i) of Definition 1. By assigning a unique key to each subset of e_0, and using the fact that the vertex list for each edge of G is sorted, it can be shown that the set E_{e_0} can be constructed in $O(r2^r|E|)$ operations. The total number of possible sets of neighbors of cardinality at most q is given by $N_q = \sum_{i=1}^{q} \binom{2^r-r-1}{i}$ and it is well-known that $N_q \leq (2^r - r)^q$. For each possible set of t neighbors for some $t \leq q$, we can check the validity of condition (ii) of Definition 1, in $O(t^2 r)$ operations. Moreover, we can check if the set $\tilde{E}$ defined in condition (iii) has the running intersection property in $O(r+t)$ operations [36]. Finally, for each valid set of neighbors, we generate the corresponding unique running intersection inequality in $\mathcal{I}^q_{e_0}$ which can be done in $O(tr)$ operations. Therefore, the total running time of the separation algorithm is given by $O(|E|(r2^r|E| + 2^{rq}q^2 r))$. □

To analyze the worst-case running time of the separation problem over all running intersection inequalities, an upper bound on the number of neighbours is needed. The following lemma provides such a bound.

Lemma 1 *Let G be a hypergraph and let $e_0 \in E(G)$. Then a running intersection inequality centered at e_0 has at most $|e_0| - 1$ neighbors.*

Proof To prove the statement, we show that each running intersection inequality centered at e_0 has at most $t-1$ neighbors, where t is the number of nodes in e_0 that are present in at least one neighbor. The lemma then follows since $t \leq |e_0|$.

By definition of a running intersection inequality, we have $t \geq 2$. Thus, the base case is $t = 2$. If $t = 2$, by conditions (i) and (ii) in definition of running intersection inequalities, we can have only one neighbor and thus the claim follows. For the induction step, we assume that $t \geq 3$. Consider a running intersection inequality centered at e_0 and let $e_1, \ldots, e_k$ be a running intersection ordering of the neighbors. If $k = 1$, then the proof is trivial. Henceforth, suppose that $k \geq 2$. We define a new running intersection inequality centered at e_0 and with neighbors $e_1, \ldots, e_{k-1}$. There exists at least one node in $e_0 \cap e_k$ that is in not contained in any other neighbor $e_1, \ldots, e_{k-1}$, as otherwise there would be an edge $e_{k'}$ among them with $e_0 \cap e_k \subseteq e_0 \cap e_{k'}$ and this contradicts the definition of a running intersection inequality. It then follows that $e_0 \cap (\bigcup_{i=1}^{k-1} e_i) \subset e_0 \cap (\bigcup_{i=1}^{k} e_i)$, and hence by the induction hypothesis the new running intersection inequality has at most $t-2$ neighbors. This implies that our original running intersection inequality has at most $t-1$ neighbors. □

The next result follows from Proposition 2 and Lemma 1.

Corollary 1 *Given a rank-r hypergraph $G = (V, E)$ and a vector $\bar{z} \in [0, 1]^{V+E}$, there exists an algorithm that solves the separation problem over all running intersection inequalities in $O(|E|(r2^r|E| + 2^{r^2} r^3))$ operations.*

Corollary 1 implies that, for a fixed-rank hypergraph $G = (V, E)$, there exists an algorithm that solves the separation problem over all running intersection inequalities in $O(|E|^2)$ operations. Before proceeding further, let us comment on the significance of the separation system by considering the use of explicit enumeration instead. Employing a similar line of analysis as in the proof of Proposition 2, we obtain an algorithm whose running time is $O(r^3|E|^r)$. It is important to note that, in almost all multilinear sets that appear in nontrivial MINLPs, we have $|E| \gg 2^r$; in fact, in many problems we have $|E| = \Theta(n^r)$. Hence, in such cases, by employing the separation system, we reduce the complexity of the separation algorithms from $O(n^{r^2})$ to $O(n^{2r})$. In addition, as we detail in the next section, the costly step associated with constructing the separation system is independent of $\bar{z}$, and hence when incorporated in a branch-and-bound algorithm, can be performed only once at the root node. Therefore, all subsequent cut generations at each node of the search tree can be carried out very efficiently.

5 Implementation in a branch-and-cut algorithm

In this section, we describe an efficient implementation of running intersection inequalities in a branch-and-cut framework. Our implementation is incorporated within the global solver BARON [22]. Our starting point is the separation algorithm described in Sect. 4. We enhance the performance of this algorithm for a branch-and-cut based solver by performing most of the expensive computations only once at the root node. More precisely, we employ the following two-step approach to generate running intersection inequalities at each node in the branch-and-cut tree:

Recognition Prior to the initialization of the branch-and-cut tree, we mark and classify all collections of multilinear equations for which running intersection inequalities can be generated later in the tree. To this end, we first identify all multilinear terms that are present in the original problem, as well as intermediate multilinear terms that are introduced via a factorable reformulation. See [2] for a formal description of the factorable reformulator in BARON. Let us demonstrate the reformulation approach by a simple example. Consider the polynomial

$$f(z) = z_1^2 z_2 z_3 + z_1^2 z_2^2 z_3 + z_1 z_2^2 z_3.$$

Obviously, this function does not contain any multilinear terms. However, a factorable reformulation of $f(z)$ is given by $f(z) = z_{234} + z_{345} + z_{135}$, where $z_{234} = z_4 z_2 z_3$, $z_{345} = z_4 z_5 z_3$, $z_{135} = z_1 z_5 z_3$, $z_4 = z_1^2$ and, $z_5 = z_2^2$. Hence, in this lifted space, we can identify a multilinear set of the hypergraph $G = (V, E)$ with $V = \{v_1, \ldots, v_5\}$ and $E = \{\{v_2, v_3, v_4\}, \{v_3, v_4, v_5\}, \{v_1, v_3, v_5\}\}$. Since factorable relaxations are already built in the lifted space, generating strong cuts for intermediate relations can significantly enhance the convergence rate of a branch-and-cut algorithm. Indeed, as we demonstrate in the next section, running intersection cuts for these lifted multilinears are quite effective for polynomial optimization problems.

Now suppose that the multilinear set $\mathcal{S}_G$ associated with a factorable reformulation of the MINLP is identified. As we detail next, at each node of the branch-and-cut tree, we solve a variant of the q-separation problem for some $q \in \{1, \ldots, r-1\}$. In order to efficiently generate the q-separation system at each node of the search tree, we construct the following data structure at the root node. Let $e_0 \in E$. For each $f \subseteq e_0$ with $|f| \geq 2$, we store the list of edges e in $E \setminus \{e_0\}$ with $e \cap e_0 = f$ and denote this list by $\mathcal{L}_{e_0}^f$. Define $\mathcal{L}_{e_0} = \{(f, \mathcal{L}_{e_0}^f) : \mathcal{L}_{e_0}^f \neq \emptyset\}$. From the proof of Proposition 2, it follows that, for a rank-r hypergraph, $\mathcal{L}_{e_0}$ can be generated in $O(r 2^r |E|)$ operations.

For each set of subsets of e_0 with $(f, \mathcal{L}^f_{e_0}) \in \mathcal{L}_{e_0}$ and of cardinality at most q, we check the validity of condition (ii) in Definition 1. We store all such sets of subsets in $\mathcal{N}_{e_0}$. It can be checked that $\mathcal{N}_{e_0}$ can be constructed in $O(rq^2 2^{rq})$ operations. Notice that it suffices to check condition (ii) only once at the root of the branch-and-cut tree. Now consider a collection of edges $e_0, e_k, k \in K$, that do not satisfy condition (iii) at the root node. Suppose that, later in the tree, a number of variables corresponding to a subset V' of nodes of G get fixed due to branching or range reduction operations; i.e., $\tilde{z}_v = 1$ for all $v \in V'$, where $\tilde{z}_v = z_v/u_v$ as defined in inequality (7). It is then possible that the collection $e_0 \backslash V'$, $e_k \backslash V'$, $k \in K$ satisfy condition (iii). Hence, to minimize the overall computational cost, we employ the following approach. At the root node, for each set F in $\mathcal{N}_{e_0}$, we check whether it has the running intersection property and, if so, we mark it so that no such a check is performed later in the tree. Otherwise, as we detail in the following, at certain nodes in the search tree, we check if F has the running intersection property. Again, by the proof of Proposition 2, this running intersection check at the root node can be performed in $O((r+q)2^{rq})$ operations. Hence, the total cost of the recognition step is given by $O(|E|(r2^r|E| + rq^2 2^{rq}))$.

Cut generation At each node in the branch-and-cut tree, we first construct and solve a crude outer-approximation of the problem based on the conventional factorable reformulation. Subsequently, various classes of cutting planes are generated and added to the current relaxation at multiple rounds, only if they violate the relaxation solution (see [41] for details). We do not add any of the running intersection inequalities to the initial outer-approximation but utilize them in the iterative cut generation scheme.

Let $\bar{z}$ denote the relaxation solution at a given cut generation round. For each $e_0 \in E$ and for each set of subsets F in $\mathcal{N}_{e_0}$, if needed, we first test if F has the running intersection property in $O(|e_0| + t)$ operations [36]. That is, this test is performed only if (i) F does not have running intersection property at the root node and (ii) at the current node, at least one variable corresponding to a node contained in an edge in F gets fixed as a result of branching or range reduction operations. Subsequently, we generate the corresponding running intersection inequality that is present in the separation system. Namely, for each $f \in F$, among all edges $e \in \mathcal{L}^f_{e_0}$, we select an edge $\bar{e}$ that minimizes $\bar{z}_e/u_e$, where u_e is the upper bound on z_e at the current node. This step can be performed in $O(|E|)$ operations. Furthermore, among all nodes $v \in N(e \cap e_0)$, we select the node that minimizes $\bar{z}_v/u_v$, where again u_v is an upper bound on z_v at the current node; this can be done in $O(q|e_0|)$ operations. We then compute the violation of this inequality by $\bar{z}$, and if the violation is larger than a preselected positive threshold, the inequality is added to BARON's cut pool. The total cost of cut generation at each round is $O(2^{rq}(|E| + rq))$.

To control the size of the LP relaxation and hence avoid any performance degradations, BARON's cut pool maintains a certain percentage of running intersection cuts with largest violation values and adds them to the relaxation of the problem at the end of each round of cut generation.

For a fixed-rank hypergraph, by employing the above two-step approach, we obtain a separation algorithm with the running time of $O(|E|^2)$ at the root node and $O(|E|)$ at each round of cut generation. As the number of cut generation rounds is often large even for moderate-sized problems, this approach leads to significant speed ups in comparison to applying the algorithm from scratch at each node of the search tree.

Finally, we should point out that for large scale problems, i.e., at the time of this writing, for problems with more than few thousands variables, the proposed recognition and cut generation schemes are quite expensive. For such problems `BARON` automatically deactivates its expensive cut generation routines, including the one proposed in this paper.

6 Numerical experiments

In this section, we demonstrate the computational benefits of incorporating running intersection cuts at every node of the branch-and-reduce global solver `BARON` [22,41]. To this end, we consider two sets of test problems. The first set contains randomly generated instances containing multilinear and polynomial optimization problems of degree three and four. The second set contains the so-called vision instances which are from an image restoration problem, a popular application in the field of computer vision. Throughout this section, all experiments are performed with `GAMS 25.1.1` on a 64-bit Intel Xeon X5650 2.66 Ghz processor; all implementations are single-threaded. In addition, all problems are solved with relative/absolute optimality tolerance of 10^{-6}, and a CPU time limit of 500 s. Other algorithmic parameters are set to the default settings of the `GAMS` distribution for all solvers. When comparing the performance of different algorithms, we call a problem trivial if all algorithms take less than half a second to solve it to optimality. All tests were performed with a development version of `BARON 18.6.19` that uses `CPLEX 12.8` [20] as the default solver for LP and MIP subproblems and performs dynamic local search using a variety of NLP solvers (see [22] for details).

6.1 Random instances

We consider a polynomial optimization problem of the form:

$$\begin{aligned} \text{(PL)} \qquad \min \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \le b_i, \qquad \forall i = 1, \dots, g \\ & x \in [0,1]^n, \end{aligned}$$

where for each $i \in \{0, 1, \dots, g\}$, $b_i \in \mathbb{R}^g$ and $f_i(x)$ is a multivariate polynomial

$$f_i(x) = \sum_{T \in \Omega_i} c_T \prod_{j \in T} x_j^{a_j},$$

Ω_i is a collection of subsets of $\{1, \dots, n\}$, while c_T, $T \in \Omega_i$ are nonzero real-valued coefficients, and the exponents a_j, $j \in T$ are positive integers. By convention, the degree of a monomial $\prod_{j \in T} x_j^{a_j}$ is the sum of its exponents, and the degree a polynomial function $f_i(x)$ is the largest degree of its monomials, i.e., $d_i = \max_T \sum_{j \in T} a_j$.

Similarly, the degree of the polynomial optimization problem (PL) is defined as the largest degree of its polynomials $d = \max_i d_i,\ i = \{0, 1, \ldots, g\}$. By letting $g = 0$, we obtain a box-constrained optimization problem. In the following, we assume that $d_i > 1$ for all $i \in \{0, 1, \ldots, g\}$.

For a polynomial f_i with n variables and degree d_i, it can be shown that the maximum number of monomial terms is given by $W_i = \sum_{k=1}^{d_i} \binom{d_i}{k}\binom{n}{k}$, where $\binom{n}{k}$ is zero if $n < k$. Denote by w_i the number of nonlinear monomials in f_i. We associate a density with each polynomial f_i defined as $\nu_i = w_i/(W_i - n)$. Accordingly, when $\nu_i = \nu$ for all $i \in \{0, 1, \ldots, g\}$, we say that the density of Problem (PL) is given by ν. Throughout this section, we characterize a polynomial optimization problem by its degree (d), number of variables (n), number of constraints (g), and density (ν). We also consider multilinear optimization problems that can be obtained by replacing the polynomial functions of Problem (PL) by multilinear functions; i.e., $a_j = 1$ for all j. For multilinear problems, we adapt the terminology defined above for polynomials, by noting that in this case we have $W_i = \sum_{k=2}^{d_i} \binom{n}{k}$. For the numerical experiments, we consider the following randomly generated test sets:

Set 1. Multilinear optimization problems of degree three with

$$(n, \nu) \in \{(10, 1.0), (15, 0.5), (20, 0.15), (20, 0.1), (25, 0.05), (30, 0.02)\},$$

and polynomial optimization problems of degree three with

$$(n, \nu) \in \{(10, 0.75), (15, 0.25), (15, 0.15), (20, 0.1), (20, 0.05)\}.$$

Set 2. Multilinear optimization problems of degree four with

$$(n, \nu) \in \{(10, 1.0), (15, 0.15), (20, 0.02), (20, 0.01), , (25, 0.01), (25, 0.005)\},$$

and polynomial optimization problems of degree four with

$$(n, \nu) \in \{(10, 0.25), (10, 0.15), (15, 0.05), (15, 0.02), (20, 0.01)\}.$$

In both test sets, we let $g \in \{0, n/5, n/2, n\}$. For each combination of $\{d, n, g, \nu\}$, we generated five problem instances, where the problem data were randomly generated from uniform distributions: the polynomial coefficients c_T were generated in the range $[-1, 1]$, while the right-hand side values b_i were generated in the range $[0, 100]$. Overall, our test set contains 220 multilinear and polynomial optimization problems of degree three, and 220 multilinear and polynomial optimization problems of degree four. This collection is obtained by adding new test problems to the test set considered in [2]. The new instances are problems with more variables and/or lower density. This augmented test set is designed to examine the effect of running intersection cuts on problems with different sparsity characteristics, ranging from boxed-constrained problem to those that are highly constrained. The test problems can be obtained from [42].

6.2 Vision instances

The vision instances are from an image restoration problem, which is widely studied in computer vision. Images are often degraded during the data acquisition process. The degradation may involve blurring, information loss due to sampling, quantization effects, and various sources of noise. The purpose of image restoration is to estimate the original image from the degraded data. We adapt the problem formulation as described in [11]. An image is a rectangle consisting of $l \times h$ pixels and it is modeled as a matrix of the same dimension where each element represents a pixel which takes value 0 or 1. An input blurred image is constructed by considering a base image and by applying a perturbation to it, that is, by changing the value of each pixel with a given probability. The image restoration model associated with a blurred image is defined as an objective function $f(x) = H(x) + L(x)$ that must be minimized. The variables x_{ij} for all $i \in \{1, \ldots, l\}$ and $j \in \{1, \ldots, h\}$ denote the value assigned to each pixel in the output image. $H(x)$ is the linear part and models similarity between the input blurred image and the output. $L(x)$ is a multilinear function of degree four and models smoothness. See [11] for a complete description of these instances. This test set consists of 45 unconstrained binary polynomial optimization problems of degree four associated with images sizes $\{10 \times 10\}$, $\{10 \times 15\}$, $\{15 \times 15\}$ with various types of base images and perturbation models.

6.3 Comparisons with the existing relaxations in BARON

We solve the test sets described in Sects. 6.1 and 6.2 to global optimality using BARON, with and without running intersection cuts. For these nonconvex problems, BARON's factorable bounds consist of supporting hyperplanes and affine envelopes for univariate monomials, along with a recursive application of bilinear envelopes to bound multilinear terms. In addition, BARON generates *multilinear cutting planes* [2] corresponding to certain facets of the convex hull of a multilinear function defined as $L(z) = \sum_{e \in E'} c_e \prod_{v \in e} z_v$, where $c_e \in \mathbb{R} \setminus \{0\}$ for all $e \in E'$ and $z_v \in [l_v, u_v]$ such that $-\infty < l_v < u_v < \infty$ for all $v \in V'$. The corresponding separation problem is an LP whose size grows exponentially with the number of variables of the multilinear function. To cope with this growth, the authors of [2] propose a decomposition technique that exploits the structure of a multilinear function to decompose it to lower-dimensional components, for which the aforementioned LP can be solved efficiently by employing a customized simplex algorithm.

To further strengthen BARON's relaxation constructor, we employ the cut generation scheme of Sect. 5 to incorporate running intersection inequalities at every node of BARON's search tree. For multilinear optimization problems, these cuts correspond to multilinear terms present in the original formulation. For polynomial optimization problems, running intersection cuts are generated for intermediate multilinears in the lifted space. In our implementation, we solve the q-separation problem described in the previous section by letting $q = \min\{r, 4\}$. Clearly, for our test sets this implies that we are solving the separation problem over all running intersection inequalities, as we have $r = 3$ and $r = 4$, in Set 1 and Set 2 of random instances, respectively, and $r = 4$ in vision instances.

Mathematically, the advantages of running intersection cuts over multilinear cuts are twofold. First, multilinear cuts correspond to facets of the convex hull of a multilinear function that appears in the objective function or a *single* constraint. However, running intersection cuts correspond to facets of multilinear polytopes where the corresponding multilinear terms may appear in multiple constraints. Hence, these cuts are particularly beneficial for constrained polynomial optimization problems. Second, the separation algorithm for multilinear cuts is significantly more expensive than that of the running intersection cuts. Hence, multilinear cuts are only generated for rather low-dimensional multilinear functions and this task is performed by heuristically decomposing the multilinear function into a collection of lower-dimensional functions. On the other hand, to generate multilinear cuts, no assumption on the structure of the underlying hypergraph is required whereas for running intersection cuts the corresponding hypergraph must satisfy conditions (i)–(iii) of Definition 1. Interestingly, as we demonstrate in the next sections, these two types of cutting planes have a complementary impact on the performance of BARON and the best results are obtained by employing a combination of the two cut generation algorithms.

To compare the performance of BARON without and with running intersection cuts, we consider the following factors: (i) execution time, (ii) total number of nodes in the branch-and-bound tree (iterations), (iii) maximum number of nodes stored in memory (nodes), and (iv) largest amount of memory utilized by BARON (memory). The memory peaks are reported in megabytes and account for all arrays allocated by BARON during the run.

6.3.1 Random instances

We first consider multilinear and polynomial optimization problems of degree three. Computational results are depicted in Fig. 1. For this test set, the number of problems that are solved to global optimality without and with running intersection cuts are 208 and 216, respectively. For a meaningful comparison, we eliminated trivial problems from the test set (35 instances). For those nontrivial problems that are solvable within 500 s by at least one of the two algorithms (181 instances), incorporating running intersection cuts in BARON results in average reductions of 60% in CPU time, 78% in number of iterations, and 70% in maximum number of nodes in memory. Finally, due to the storage of the running-intersection data structure at the recognition step, there is a modest increase in memory requirements.

The results of Fig. 1 are analyzed in Table 1a to further quantify the effect of incorporating the running intersection cuts. The first line of Table 1a provides the percentage of nontrivial problems for which running intersection cuts lead to at least a factor of two improvement with respect to CPU time, number of iterations, and number of nodes in memory. The subsequent lines of the table provides similar statistics for problems for which the algorithm was improved by at least 30% but no more than 50%, problems for which there was no significant performance change after addition of cutting planes, and problems for which there was some deterioration in performance because of cutting planes. As can be seen from Table 1a, for about two third of the

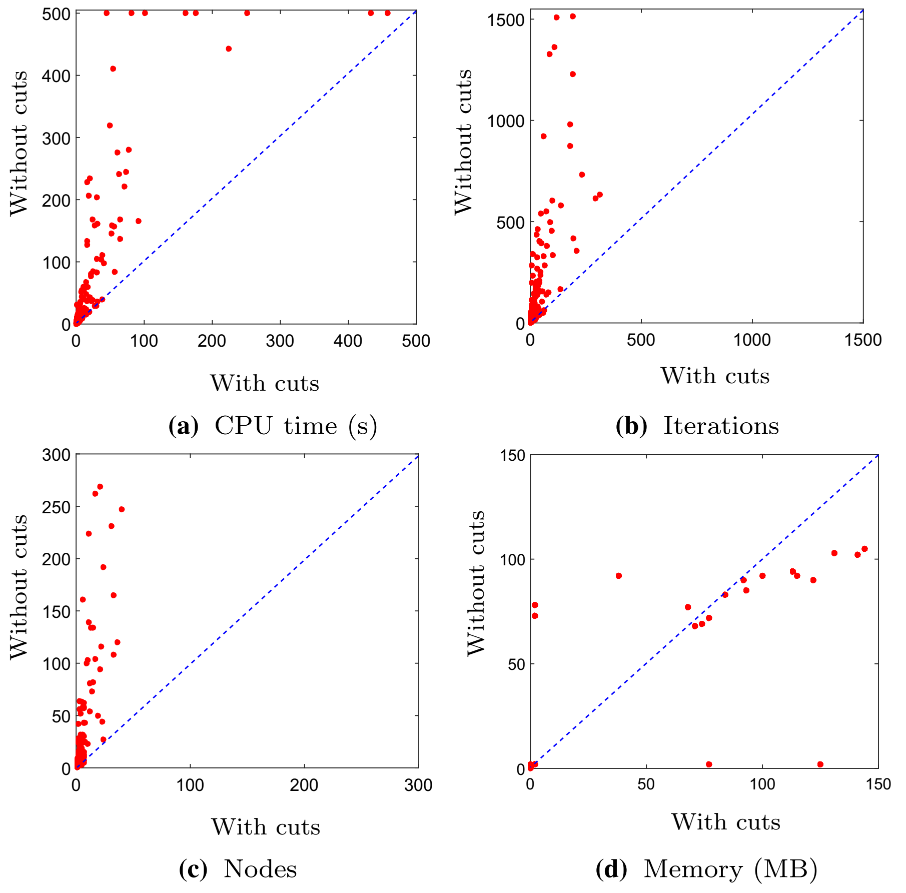


Fig. 1 Performance of BARON with and without running intersection cuts for 220 third-degree multilinear and polynomial optimization problems. In these figures, nontrivial problems that are solved in less than 500 s are compared with respect to **a** CPU time, **b** number of iterations, **c** maximum number of nodes in memory and, **d** largest amount of memory allocated

test problems, running intersection cuts reduce the CPU time of the global solver by at least a factor of two. Similar improvements are observed with respect to the total number and maximum number of nodes in the branch-and-cut tree. We observe minor performance degradations for about 3% of the test problems, most of which are due to solving more expensive LP relaxations obtained as a result of the addition of running intersection cuts.

Next, we examine the impact of running intersection cuts on multilinear and polynomial optimization problems of degree four. Out of 220 problems, 181 instances are solved to global optimality within 500 s by at least one of the two algorithms, 22 of which are trivial. The number of solvable problems to global optimality without and with running intersection cuts are 168 and 180, respectively. As can be seen from Fig. 2 and Table 1b, the proposed cuts significantly improve the performance of BARON for this test set. Specifically, we obtain average reductions of 43% in CPU

Table 1 Effect of adding running intersection cuts to BARON for random instances

Effect of adding cuts	CPU time	Iterations	Nodes
(a) Third-degree multilinear and polynomial optimization problems			
Better by a factor at least 2	113 (62%)	130 (72%)	129 (71%)
Between 30% and 100% better	21 (12%)	18 (10%)	14 (8 %)
Difference smaller than 30%	42 (23%)	31 (17%)	36 (20%)
Between 30% and 100% worse	5 (3%)	2 (1%)	2 (1%)
Worse by a factor of at least 2	0 (0%)	0 (0%)	0 (0%)
(b) Fourth-degree multilinear and polynomial optimization problems			
Better by a factor at least 2	55 (35%)	114 (72%)	113 (71%)
Between 30% and 100% better	19 (12%)	16 (10%)	16 (10%)
Difference smaller than 30%	59 (37%)	25 (16%)	27 (17%)
Between 30% and 100% worse	16 (10%)	0 (0%)	0 (0%)
Worse by a factor of at least 2	10 (6%)	4 (2%)	3 (2%)

time, 76% in number of iterations, and 72% in maximum number of nodes in memory. For this test set, the increase in the largest amount of allocated memory is more significant as the size of the data structure allocated in the recognition step is increasing in the degree of the multilinear set. Table 1b indicates that, thanks to the running intersection cuts, for more than one third of the problems, the execution time is improved by at least a factor of two. However, we also observe some notable performance degradations for about 6% of the problems. Again, this adverse effect is due to the increase in the size of the LP relaxations. For higher degree problems, many more running intersection inequalities can be generated and while these cuts improve the quality of the lower bounds in general, they may increase the overall cost of the branch-and-bound tree as the resulting LPs are often much more expensive to solve.

6.3.2 Vision instances

This test set contains 45 optimization problems none of which are solved trivially and only one instance cannot be solved to global optimality within 500 s by either of the two algorithms. The number of solvable problems to global optimality without and with running intersection cuts are 37 and 44, respectively. By employing the proposed cutting planes we obtain average reductions of 63% in CPU time, 42% in number of iterations, and 30% in maximum number of nodes in memory. The computational benefits of running intersection cuts on vision instances are further detailed in Table 2. As can be seen from this table, thanks to the running intersection cuts, for about 40% of the problems, the execution time is improved by at least a factor of two. In fact, by employing the proposed cutting planes, BARON is able to solve all 44 vision instances at the root node of the branch-and-bound tree. Moreover, for this test set, the increase in the largest amount of allocated memory is negligible.

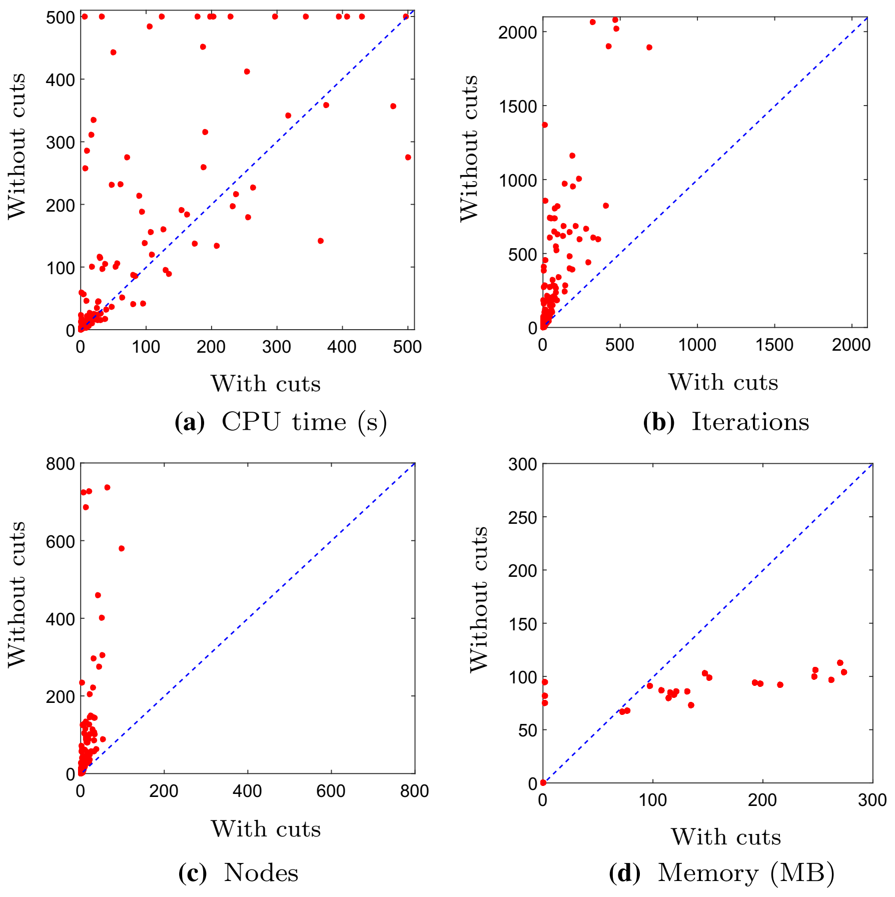
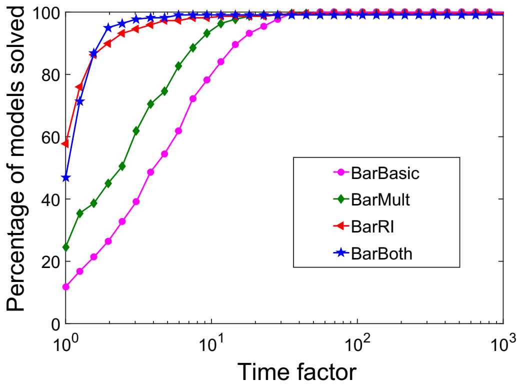


Fig. 2 Performance of BARON with and without running intersection cuts for 220 fourth-degree multilinear and polynomial optimization problems. In these figures, nontrivial problems that are solved in less than 500 s are compared with respect to **a** CPU time, **b** number of iterations, **c** maximum number of nodes in memory and, **d** largest amount of memory allocated

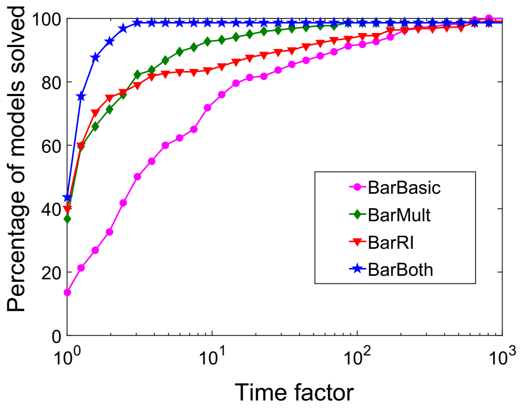
Table 2 Effect of adding running intersection cuts to BARON for vision instances

Effect of adding cuts	CPU time	Iterations	Nodes
Better by a factor at least 2	17 (38%)	10 (23%)	10 (23%)
Between 30% and 100% better	13 (30%)	0 (0%)	0 (0%)
Difference smaller than 30%	14 (32%)	34 (77%)	34 (77%)
Between 30% and 100% worse	0 (0%)	0 (0%)	0 (0%)
Worse by a factor of at least 2	0 (0%)	0 (0%)	0 (0%)

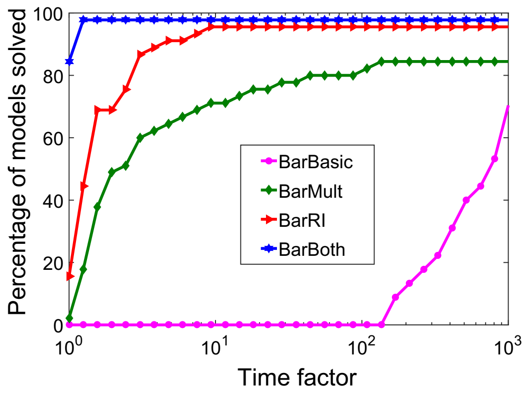
Fig. 3 Impact of different cut generation strategies on BARON for **a** 220 randomly generated multilinear and polynomial optimization problems of degree three, **b** 220 randomly generated multilinear and polynomial optimization problems of degree four, and **c** 45 computer vision problems



(a) Third degree problems



(b) Fourth degree problems



(c) Computer vision instances

Table 3 Comparison of CPU times (geometric means) using different cut generation schemes

Test set	BarBasic	BarMult	BarRI	BarBoth
Third degree	12.75	6.60	3.42	3.02
Fourth degree	77.49	23.36	30.55	15.25
Computer vision	495.65	8.97	5.04	3.36

6.4 Relative impact of multilinear cuts versus running intersection cuts

The only other type of cutting planes in BARON which are generated for multilinear sets are multilinear cuts, as introduced in [2]. In this section, we examine the relative impact of multilinear cuts and running intersection cuts. Results in [2] indicate that multilinear cuts significantly enhance the performance of BARON. In the previous section, we demonstrated that the addition of running intersection cuts leads to further improvements. The purpose of this section is to understand the relative impact of the two cut generation strategies on different problem types. To this end, we consider the following four versions of BARON:

1. BarBasic No multilinear cuts or running intersection cuts are generated; in this version, for our test sets, BARON's relaxation constructor is essentially a conventional term-wise factorable scheme [41].
2. BarMult Multilinear cuts are generated but no running intersection cut is generated.
3. BarRI Running intersection cuts are generated but no multilinear cuts are generated.
4. BarBoth Both multilinear cuts and running intersection cuts are generated.

To evaluate the performance of BARON with different cut generation strategies, we make use of performance profiles [16]. The performance profile of a solver is a (cumulative) distribution function for a performance metric. Throughout this section, we use the ratio of the time that an algorithm takes to solve a problem versus the best time of all algorithms as the performance metric. For the purpose of constructing performance profiles, by *solve* we imply that the algorithm finds the best solution among all solvers for a given problem within the time limit. We utilize the GAMS performance tools [18] to construct all performance profiles. To account for trivial problems as we described in the previous section, we set resmin= 0.5 s. As defined by the GAMS performance tools manual, resmin is the minimum resource time threshold. That is, if a solver reports a resource time below resmin, then it is increased to resmin before being used in ratio calculations.

Figure 3 shows the performance profiles of the four variants of BARON for all test sets. As can be seen from this figure, for all test sets BarBoth is the best solver whereas BarBasic is the worst. The relative performance of BarMult and BarRI however changes in the two random test sets. For the computer vision test set, the relative impact of running intersection cuts is more significant than that of the multilinear cuts. In Table 3, for each cut generation strategy, we report the geometric mean of the CPU time taken over each test; for all test tests, BarBoth is on average the fastest solver.

Hence, we conclude that the combination of multilinear cuts and running intersection cuts outperforms either of the two cut generation schemes in isolation.

We conclude by acknowledging that for a general polynomial optimization problem, the impact of running intersection cuts on the convergence rate of the branch-and-bound tree highly depends on the structure of the underlying hypergraph. In particular, we have observed that for problems corresponding to highly dense hypergraphs, the proposed cuts may not lead to any visible improvements. See for example the auto-correlation test set presented in [29].

References

1. Anthony, M., Boros, E., Crama, Y., Gruber, A.: Quadratic reformulations of nonlinear binary optimization problems. Math. Program. **162**(1), 115–144 (2017). https://doi.org/10.1007/s10107-016-1032-4
2. Bao, X., Khajavirad, A., Sahinidis, N.V., Tawarmalani, M.: Global optimization of nonconvex problems with multilinear intermediates. Math. Program. Comput. **7**, 1–37 (2015)
3. Barahona, F., Jünger, M., Reinelt, G.: Experiments in quadratic $0-1$ programming. Math. Program. **44**, 127–137 (1989)
4. Beeri, C., Fagin, R., Maier, D., Yannakakis, M.: On the desirability of acyclic database schemes. J. ACM **30**, 479–513 (1983)
5. Belotti, P., Lee, J., Liberti, L., Margot, F., Wächter, A.: Branching and bounds tightening techniques for non-convex MINLP. Optim. Methods Softw. **24**, 597–634 (2009)
6. Berthold, T., Gamrath, G., Hendel, G., Heinz, S., Koch, T., Pfetsch, M., Vigerske, S., Waniek, R., Winkler, M., Wolter, K.: SCIP 3.2, User's Manual (2016)
7. Bienstock, D., Munoz, G.: LP furmulations for polynomial optimization problems. SIAM J. Optim. **28**(2), 1121–1150 (2018)
8. Bonami, P., Günlük, O., Linderoth, J.: Globally solving nonconvex quadratic programming problems with box constraints via integer programming methods. Math. Program. Comput. (2018). https://doi.org/10.1007/s12532-018-0133-x
9. Buchheim, C., Rinaldi, G.: Efficient reduction of polynomial zero-one optimization to the quadratic case. SIAM J. Optim. **18**, 1398–1413 (2007)
10. Crama, Y.: Concave extensions for non-linear $0-1$ maximization problems. Math. Program. **61**, 53–60 (1993)
11. Crama, Y., Rodríguez-Heck, E.: A class of valid inequalities for multilinear $0-1$ optimization problems. Discrete Optim. **25**, 28–47 (2017)
12. Del Pia, A., Khajavirad, A.: A polyhedral study of binary polynomial programs. Math. Oper. Res. **42**(2), 389–410 (2017)
13. Del Pia, A., Khajavirad, A.: On decomposability of multilinear sets. Math. Program. (2017). https://doi.org/10.1007/s10107-017-1158-z
14. Del Pia, A., Khajavirad, A.: The multilinear polytope for acyclic hypergraphs. SIAM J. Optim. **28**, 1049–1076 (2018)
15. Del Pia, A., Khajavirad, A.: The running intersection relaxation of the multilinear polytope. Optim. Online manuscript 2018/05/6618 (2018)
16. Dolan, E., More, J.: Benchmarking optimization software with performance profiles. Math. Program. **91**, 201–213 (2002)
17. Fix, A., Gruber, A., Boros, E., Zabih, R.: A graph cut algorithm for higher-order Markov random fields. In: 2011 International Conference on Computer Vision, pp. 1020–1027 (2011). https://doi.org/10.1109/ICCV.2011.6126347
18. GAMS Performance tools. Available at http://www.gams.com/help/topic/gams.doc/solvers/allsolvers.pdf
19. Helmberg, C., Rendl, F.: Solving quadratic $0-1$ problems by semidefinite programs and cutting planes. Math. Program. **82**, 291–315 (1998)
20. IBM: CPLEX Optimizer (2016). http://www-01.ibm.com/software/integration/optimization/cplex-optimizer/

21. Karp, R.M.: Reducibility among combinatorial problems. In: Millera, R.E., Thatcher, J.W. (eds.) Complexity of Computer Computations. New York (1972)
22. Khajavirad, A., Sahinidis, N.V.: A hybrid LP/NLP paradigm for global optimization relaxations. Math. Program. Comput. (2018). https://doi.org/10.1007/s12532-018-0138-5
23. Lin, Y., Schrage, L.: The global solver in the LINDO API. Optim. Methods Softw. **24**, 657–668 (2009)
24. McCormick, G.P.: Computability of global solutions to factorable nonconvex programs: part I-convex underestimating problems. Math. Program. **10**, 147–175 (1976)
25. Meyer, C.A., Floudas, C.A.: Trilinear monomials with positive or negative domains: facets of the convex and concave envelopes. In: Floudas, C.A., Pardolos, P.M. (eds.) Frontiers in Global Optimization, vol. 103, pp. 327–352. Kluwer Academic Publishers, Norwell (2003)
26. Meyer, C.A., Floudas, C.A.: Trilinear monomials with mixed sign domains: facets of the convex and concave envelopes. J. Glob. Optim. **29**, 125–155 (2004)
27. Misener, R., Floudas, C.A.: ANTIGONE: algorithms for continuous/integer global optimization of nonlinear equations. J. Glob. Optim. **59**, 503–526 (2014)
28. Padberg, M.: The boolean quadric polytope: some characteristics, facets and relatives. Math. Program. **45**, 139–172 (1989)
29. POLIP: Library for polynomially constrained mixed-integer programming (2014). http://polip.zib.de
30. Rikun, A.D.: A convex envelope formula for multilinear functions. J. Glob. Optim. **10**, 425–437 (1997)
31. Ryoo, H.S., Sahinidis, N.V.: Analysis of bounds for multilinear functions. J. Glob. Optim. **19**, 403–424 (2001)
32. Sahinidis, N.: Sahinidis optimization group website. http://archimedes.cheme.cmu.edu/?q=baron
33. Schrijver, A.: Theory of Linear and Integer Programming. Wiley, Chichester (1986)
34. Sherali, H.D.: Convex envelopes of multilinear functions over a unit hypercube and over special discrete sets. Acta Math. Vietnam. **22**, 245–270 (1997)
35. Sherali, H.D., Adams, W.P.: A hierarchy of relaxations between the continuous and convex hull representations for zero-one programming problems. SIAM J. Discrete Math. **3**, 411–430 (1990)
36. Tarjan, R.E., Yannakakis, M.: Simple linear-time algorithms to test chordality of graphs, test acyclicity of hypergraphs, and selectively reduce acyclic hypergraphs. SIAM J. Comput. **13**(3), 566–579 (1984)
37. Tawarmalani, M.: Inclusion certificates and simultaneous convexification of functions. Working paper (2010). http://www.optimization-online.org/DB_FILE/2010/09/2722.pdf
38. Tawarmalani, M., Richard, J.P., Chung, K.: Strong valid inequalities for orthogonal disjunctions and bilinear covering sets. Math. Program. **124**, 481–512 (2010)
39. Tawarmalani, M., Richard, J.P., Xiong, C.: Explicit convex and concave envelopes through polyhedral subdivisions. Math. Program. **138**, 531–577 (2013)
40. Tawarmalani, M., Sahinidis, N.V.: Global optimization of mixed-integer nonlinear programs: a theoretical and computational study. Math. Program. **99**, 563–591 (2004)
41. Tawarmalani, M., Sahinidis, N.V.: A polyhedral branch-and-cut approach to global optimization. Math. Program. **103**, 225–249 (2005)
42. The Optimization Firm, LLC: NLP and MINLP test problems. https://minlp.com/nlp-and-minlp-test-problems
43. Yajima, Y., Fujie, T.: A polyhedral approach for nonconvex quadratic programming problems with box constraints. J. Glob. Optim. **13**, 151–170 (1998)

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